A PERTURBATION THEOREM OF MIYADERA TYPE FOR LOCAL
C -REGULARIZED SEMIGROUPS

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Abstract. In this paper, we investigate the perturbation problem for local
C-regularized semigroups on a Banach space and establish a Miyadera type
perturbation theorem.

1. INTRODUCTION AND PRELIMINARIES

The Miyadera perturbation theorem for C₀ semigroups was established in 1966
([4]). Since then, there have been some generalizations (cf., e.g., [1, 3, 5, 9] and
references therein). The aim of this paper is to extend this theorem to local C-
regularized semigroups (introduced in [8]) and present a Miyadera type perturbation
theorem. This result contains the classical Miyadera perturbation theorem as a
special case. Moreover, it is also suitable for non-exponentially-bounded regularized
semigroups, while the C₀ semigroup and the other operator families concerned in
[1, 3, 5, 9] are all exponentially bounded on [0, ∞). For more information on local
regularized semigroups and regularized semigroups, we refer the reader to [2, 6, 7,
8, 10] and references cited there.

Throughout this paper, all operators are linear; X is a Banach space; \( L(X, Y) \)
denotes the space of all continuous linear operators from X to a space Y, and
\( L(X, X) \) will be abbreviated to \( L(X) \); \( L_s(X) \) is the space of all continuous linear
operators from X to X with the strong operator topology; \( C \) is an injective operator
in \( L(X) \); \( C([0, t], L_s(X)) \) is the space of all strongly continuous \( L(X) \)-valued
functions, equipped with the norm

\[
\| F \|_\infty = \sup_{r \in [0, t]} \| F(r) \|.
\]
Moreover, for an operator $A$, we write $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\rho(A)$, respectively, for the domain, the range, the resolvent set of $A$, and we denote by $[\mathcal{D}(A)]$ the space $\mathcal{D}(A)$ with the graph norm.

**Definition 1.1.** ([8]) Assume $\tau > 0$. A one-parameter family $\{T(t)\}_{t \in [0, \tau]} \subset \mathcal{L}(X)$ is called a local $C$-regularized semigroup on $X$ if

(i) $T(0) = C$ and $T(t + s)C = T(t)T(s)$ \quad (\forall s, t, s + t \in [0, \tau]),

(ii) $T(\cdot)x : [0, \tau] \rightarrow X$ is continuous for every $x \in X$.

The operator $A$ defined by

$$
\mathcal{D}(A) = \{ x \in X : \lim_{t \to 0^+} \frac{1}{t}(T(t)x - Cx) \text{ exists and is in } \mathcal{R}(C) \}
$$

and

$$
Ax = C^{-1} \lim_{t \to 0^+} \frac{1}{t}(T(t)x - Cx), \quad \forall x \in \mathcal{D}(A),
$$

is called the generator of $\{T(t)\}_{t \in [0, \tau]}$. It is also called that $A$ generates $\{T(t)\}_{t \in [0, \tau]}$.

**Remark 1.2.** When $C = I$, $\{T(t)\}_{t \in [0, \tau]}$ can be extended uniquely (in an obvious way) to a $C_0$ semigroup $\{T(t)\}_{t \geq 0}$ with $A$ as its generator.

The following two lemmas will be used freely in the proofs of our results below. Lemma 1.3 comes from [8] and Lemma 1.4 is implied in [2].

**Lemma 1.3.** Let $A$ generate a local $C$-regularized semigroup $\{T(t)\}_{t \in [0, \tau]}$ on $X$. Then

(i) For $x \in \mathcal{D}(A)$, $t \in [0, \tau]$, $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$.

(ii) For $x \in X$, $t \in [0, \tau]$, $\int_0^t T(s)xds \in \mathcal{D}(A)$ and $A\int_0^t T(s)xds = T(t)x - Cx$.

(iii) For $x \in \mathcal{D}(A)$, $t \in [0, \tau]$, $\int_0^t T(s)Axds = A\int_0^t T(s)xds = T(t)x - Cx$.

**Lemma 1.4.** Suppose an extension of $A$, $\tilde{A}$, generates a local $C$-regularized semigroup. Then $C(\mathcal{D}(\tilde{A})) \subset \mathcal{D}(A)$ is equivalent to $C^{-1}AC = \tilde{A}$.

2. Results and Proofs

**Theorem 2.1.** Assume that a densely defined linear operator $A$ in $X$ generates a local $C$-regularized semigroup $\{T(t)\}_{t \in [0, \tau]}$ on $X$. If $P \in \mathcal{L}(X)$ satisfying

(H1) $\rho((I + P)A) \neq \emptyset$,
(H2) for all \( x \in D(A) \), and \( \Psi \in C([0, \tau], L_s(X)) \),
\[
\left\| \int_0^t \Psi(s)C^{-1} PAT(t-s)xds \right\| \leq \beta(t)\|\Psi\|_\infty\|x\|, \quad t \in [0, \tau],
\]
where \( \beta(\cdot) \) is a function with \( \lim \sup_{t \to 0^+} \beta(t) < 1 \),

(H3) there exists an injective operator \( C_1 \in L(X) \) such that \( \mathcal{R}(P) \subset \mathcal{R}(C_1) \subset \mathcal{R}(C) \), \( C_1(I + P)A \subset (I + P)AC_1 \), and \( C_1^{-1}(D(A)) \) is a dense subspace in \( D(A) \),

then \( (I + P)A \) generates a local \( C_1 \)-regularized semigroup on \( X \).

**Proof.** Let \( \tau > \tau_1 > 0 \), such that \( \beta(t) \leq \kappa < 1 \), for all \( t \in [0, \tau_1] \). Define
\[
(H\mathcal{U})(t)x = \int_0^t \mathcal{U}(s)C^{-1} PAT(t-s)xds, \quad t \in [0, \tau_1], \ x \in D(A),
\]
for any strongly continuous operator function \( \mathcal{U} : [0, \tau_1] \to L(X) \).

Clearly, \( (H\mathcal{U})(t)x \) is continuous in \( t \) on \( [0, \tau_1] \) and depends linearly on \( x \in D(A) \). Since
\[
\| (H\mathcal{U})(t)x \| = \left\| \int_0^t \mathcal{U}(s)C^{-1} PAT(t-s)xds \right\|
\leq \beta(t)\|\mathcal{U}\|_\infty\|x\|
\]
for every \( t \in [0, \tau_1] \), and \( D(A) \) is dense in \( X \), we can extend the operator \( (H\mathcal{U})(t) \) to a continuous operator on \( X \), and the extended operator function \( (\overline{H\mathcal{U}})(\cdot) \) is strongly continuous on \( [0, \tau_1] \). Hence \( H \) maps \( C([0, \tau_1], L_s(X)) \) into itself. Since
\[
\|(\overline{H\mathcal{U}}_1 - \overline{H\mathcal{U}}_2)(t)\| \leq \beta(t)\|\mathcal{U}_1 - \mathcal{U}_2\|_\infty \leq \kappa\|\mathcal{U}_1 - \mathcal{U}_2\|_\infty,
\]
there exists a unique \( \mathcal{U} \in C([0, \tau_1], L_s(X)) \) satisfying
\[
(2.1) \quad \mathcal{U}(t)x = T(t)x + \int_0^t \mathcal{U}(s)C^{-1} PAT(t-s)xds, \quad t \in [0, \tau_1], \ x \in D(A).
\]
Setting
\[
\mathcal{V}(t) = \mathcal{U}(t)C^{-1} C_1, \quad t \in [0, \tau_1],
\]
we have, from (2.1),
\[
\mathcal{V}(t)x = T(t)C^{-1} C_1x + \int_0^t \mathcal{V}(s)C_1^{-1} PAT(t-s)C_1^{-1} C_1xds, \quad x \in D(A), \ t \in [0, \tau_1].
\]
Hence, for $x \in \mathcal{D}(A)$,
\[
\int_0^t \mathcal{V}(s) x ds = \int_0^t T(s)C^{-1}C_1 x ds \\
+ \int_0^t \int_0^s \mathcal{V}(\sigma)C_1^{-1}PAT(s-\sigma)C^{-1}C_1 x d\sigma ds,
\quad t \in [0, \tau_1].
\]
It follows that for $x \in X$,
\begin{align*}
\int_0^t \mathcal{V}(s) x ds &= \int_0^t T(s)C^{-1}C_1 x ds \\
&\quad + \int_0^t \mathcal{V}(\sigma)C_1^{-1}P[T(t-\sigma)C^{-1}C_1 x - C^{-1}C_1 x] d\sigma, 
\quad t \in [0, \tau_1],
\end{align*}
(2.2)
due to the density of $\mathcal{D}(A)$. Note $\mathcal{D}(A) \subset \mathcal{D}(C_1^{-1}PAC_1)$, since $AC_1 = (AC)(C_1^{-1}C_1)$ and $C_1^{-1}C_1$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$. So, for $x \in \mathcal{D}(A)$,
\[
\int_0^t \mathcal{V}(s)C_1^{-1}PAC_1 x ds = \int_0^t T(s)C_1^{-1}PAC_1 x ds \\
+ \int_0^t \mathcal{V}(\sigma)C_1^{-1}P[T(t-\sigma)C_1^{-1}PAC_1 x - PAC_1 x] d\sigma,
\]
by (2.2). Thus, we see that for $x \in \mathcal{D}(A)$,
\[
\int_0^t \mathcal{V}(s)(I + P)Ax ds \\
= \int_0^t T(s)C^{-1}(I + P)AC_1 x ds \\
+ \int_0^t \mathcal{V}(\sigma)C_1^{-1}P[T(t-\sigma)C^{-1}(I + P)AC_1 x - (I + P)AC_1 x] d\sigma \\
(2.3)
= T(t)C_1 x - C_1 x + \int_0^t T(s)C^{-1}PAC_1 x ds \\
+ \int_0^t \mathcal{V}(\sigma)C_1^{-1}P[T(t-\sigma)C^{-1}_1 x ds - \int_0^t \mathcal{V}(\sigma)C_1^{-1}PAC_1 x ds \\
+ \int_0^t \mathcal{V}(\sigma)C_1^{-1}P[T(t-\sigma)C^{-1}_1 PAC_1 x - PAC_1 x] d\sigma \\
= \mathcal{V}(t)x - C_1 x.
\]
Now we consider the integral equation
\[
(2.4) \quad h(t)x = C_1 x + \int_0^t h(s)(I + P)Ax ds, \quad x \in \mathcal{D}(A), \quad t \in [0, \tau_1],
\]
for \( h(t) \in C([0, \tau_1], \mathcal{L}_s(X)) \). Let \( h(t) \) be a solution of (2.4). Then from (2.4) it follows that for \( x \in \mathcal{D}(A) \),
\[
\int_0^t h(s)(I + P)A \int_0^{t-s} T(\sigma)C^{-1}C_1 x d\sigma ds \quad = \int_0^t \int_0^{t-s} h(s)(I + P)AT(\sigma)C^{-1}C_1 x d\sigma d\sigma \\
= \int_0^t h(s)T(t \cdot s)C^{-1}C_1 x ds - C_1 \int_0^t T(s)C^{-1}C_1 x ds.
\]
On the other hand, for \( x \in \mathcal{D}(A) \),
\[
\int_0^t h(s)(I + P)A \int_0^{t-s} T(\sigma)C^{-1}C_1 x d\sigma ds \\
= \int_0^t [T(t \cdot s)C^{-1}C_1 x - C_1 x] ds + \int_0^t h(s)PA \int_0^{t-s} T(\sigma)C^{-1}C_1 x d\sigma ds.
\]
Hence, for \( x \in \mathcal{D}(A) \),
\[
\int_0^t h(s)C_1 x ds = C_1 \int_0^t T(s)C^{-1}C_1 x ds + \int_0^t h(s)PA \int_0^{t-s} T(\sigma)C^{-1}C_1 x d\sigma ds,
\]
that is,
\[
(h(t)C)C^{-1}C_1 x = C_1 T(t)C^{-1}C_1 x + \int_0^t (h(s)C)C^{-1}PAT(t \cdot s)C^{-1}C_1 x d\sigma.
\]
Noting that \( C^{-1}C_1(\mathcal{D}(A)) \subset \mathcal{D}(A) \) is dense in \( X \), and the solution \( \overline{h}(t) \) of the equation
\[
\overline{h}(t)y = C_1 T(t)y + \int_0^t \overline{h}(s)C^{-1}PAT(t \cdot s)y ds, \quad y \in C^{-1}C_1(\mathcal{D}(A)), \quad t \in [0, \tau_1]
\]
in \( C([0, \tau_1], \mathcal{L}_s(X)) \) is unique, we see the solution of (2.4) is also unique.

By the equality (2.3), (H3), the uniqueness of solution of (2.4) and the density of \( \mathcal{D}(A) \), we obtain
\[
(\lambda_0 - (I + P)A)^{-1}V(t) = V(t)(\lambda_0 - (I + P)A)^{-1}, \quad t \in [0, \tau_1], \quad \lambda_0 \in \rho((I + P)A),
\]
and therefore
\[
(I + P)A V(t)x = V(t)(I + P)A x, \quad x \in \mathcal{D}(A), \quad t \in [0, \tau_1].
\]
Since \( \rho((I + P)A) \neq \emptyset \), \((I + P)A\) is a closed operator. Thus from (2.3), the denseness of \( D(A) \) and the closedness of \((I + P)A\), it follows that \( \int_0^t V(s)xds \in D(A) \) and

\[
V(t)x = C_1x + (I + P)A \int_0^t V(s)xds, \quad x \in X, \quad t \in [0, \tau_1].
\]

Let \( x \in D(A) \). Then for \( t, h \in [0, \tau_1] \),

\[
\begin{align*}
V(h)V(t)x &= V(h) \int_0^t V(\sigma)(I + P)Ax d\sigma + V(h)C_1x \\
&= \int_0^t V(h)V(\sigma)(I + P)Ax d\sigma + C_1^2x + \int_0^h V(s)(I + P)AC_1xds,
\end{align*}
\]

and that for \( t, t + h \in [0, \tau_1] \),

\[
\begin{align*}
V(t + h)C_1x &= \int_0^{t+h} V(s)(I + P)AC_1xds + C_1^2x \\
&= \int_h^{t+h} V(s)(I + P)AC_1xds + \int_0^h V(s)(I + P)AC_1xds + C_1^2x \\
&= \int_0^t V(s + h)C_1(I + P)Axds + C_1^2x + \int_0^h V(s)(I + P)AC_1xds.
\end{align*}
\]

As a consequence,

\[
V(h)V(t)x - V(h + t)C_1x = \int_0^t [V(h)V(\sigma) - V(\sigma + h)C_1](I + P)Ax d\sigma.
\]

It follows from the uniqueness of the solution of (2.4) that

\[
V(t)V(h) = V(t + h)C_1, \quad t, h, t + h \in [0, \tau_1].
\]

Hence \( \{V(t)\}_{t \in [0, \tau_1]} \) is a local \( C_1 \)-regularized semigroup on \( X \). Denote by \( A_0 \) the generator of \( \{V(t)\}_{t \in [0, \tau_1]} \). We see easily from (2.3) that \( D((I + P)A) = D(A) \subset D(A_0) \). On the other hand, for any \( x \in D(A_0) \), we have

\[
\lim_{m \to \infty} m \int_0^\frac{1}{m} V(s)xds = C_1x,
\]

\[
\lim_{m \to \infty} (I + P)A \left[ m \int_0^\frac{1}{m} V(s)xds \right] = \lim_{m \to \infty} m \left[ V \left( \frac{1}{m} \right) x - C_1x \right] = C_1A_0x.
\]
by (2.5). It follows that $C_1(\mathcal{D}(A_0)) \subset \mathcal{D}((I + P)A)$. Consequently, $A_0 = C_1^{-1}(I + P)AC_1$ by Lemma 1.4. But

$$C_1^{-1}(I + P)AC_1 = (I + P)A,$$

since $\rho((I + P)A) \neq \emptyset$. This ends the proof.

**Corollary 2.2.** Suppose that a densely defined linear operator $A$ in $X$ generates a local $C$-regularized semigroup $\{T(t)\}_{t \in [0, \tau]}$ on $X$. If $B \in \mathcal{L}([\mathcal{D}(A)], X)$ satisfying

(H1') $\rho(A) \neq \emptyset$ and $\rho(A + B) \neq \emptyset$,

(H2') there exist $\tau_1 \in (0, \tau], \gamma \in (0, 1)$ such that

$$\int_0^{\tau_1} \|C^{-1} BT(s)x\| \, ds \leq \gamma \|x\|, \quad x \in \mathcal{D}(A),$$

(H3') there exists an injective operator $C_1 \in \mathcal{L}(X)$ such that $\mathcal{R}(B) \subset \mathcal{R}(C_1) \subset \mathcal{R}(C)$, $C_1(A + B) \subset (A + B)C_1$, and $C^{-1}C_1(\mathcal{D}(A))$ is a dense subspace in $\mathcal{D}(A)$,

then $A + B$ generates a local $C_1$-regularized semigroup.

**Proof.** Take $\lambda_0 \in \rho(A)$. Then $A - \lambda_0$ generates a local $C$-regularized semigroup $\{e^{-\lambda_0t}T(t)\}_{t \in [0, \tau]}$ on $X$. Setting $P = B(A - \lambda_0)^{-1}$, we have $P \in \mathcal{L}(X)$. It’s clear from (H2') that for $x \in \mathcal{D}(A)$, and $\Psi \in C([0, \tau_1], \mathcal{L}_u(X))$,

$$\left\| \int_0^t \Psi(s)C^{-1} P(A - \lambda_0)e^{-\lambda_0(t-s)}T(t-s)x \, ds \right\| \leq \gamma_1 \|\Psi\|_\infty \|x\|, \quad t \in [0, \tau_1],$$

for some $\tau_1 \in (0, \tau], \gamma_1 \in (\gamma, 1)$. Thus making use of Theorem 2.1, we infer that $(I + P)(A - \lambda_0)$ generates a local $C_1$-regularized semigroup $\{V(t)\}_{t \in [0, \tau_1]}$, and therefore $A + B = (I + P)(A - \lambda_0) + \lambda_0$ is the generator of the local $C_1$-regularized semigroup $\{e^{\lambda_0t}V(t)\}_{t \in [0, \tau_1]}$. This completes the proof. 

**Remark 2.3.** Corollary 2.2 is a generalization of the Miyadera perturbation theorem ([4]). Actually, when $A$ generates a $C_0$ semigroup on $X$, and $C = C_1 = I$, Corollary 2.2 is just the Miyadera perturbation theorem (see also Remark 1.2).

Finally, we present a concrete example to show how our results can be used.

**Example 2.4.** Let $X_1 = L^2(\Omega)$, $X_2 = C_0(\gamma)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and

$$\gamma := \{s + ie^{s^2}; \ s \geq 0\}.$$
Define
\[ A_1 := i\Delta, \quad \mathcal{D}(A_1) = H^2(\Omega) \cap H_0^1(\Omega), \]
\[ (A_2\varphi)(\xi) = \xi\varphi(\xi), \quad \text{with } \varphi \in \mathcal{D}(A_2) := \{ \varphi \in C_0(\gamma); \xi \mapsto \xi\varphi(\xi) \in C_0(\gamma) \}. \]

Then, \( A_1 \) generates a strongly continuous group \( \{ T_1(t) \}_{t \in \mathbb{R}} \) on \( X_1, \mathcal{D}(A_2) = X_2 \) and \( A_2 \) generates (cf. [2, p. 110, Ex. 18.2]) an \( A_{-2}^{-1} \)-regularized semigroup \( \{ T_2(t) \}_{t \geq 0} \) on \( X_2 \) given by
\[ T_2(t)\varphi(\xi) = \frac{1}{\xi} e^{\xi t}\varphi(\xi). \]

Let \( q_1, q_2 \in C_c(\Omega), r_1 \in \mathcal{D}(A_1), r_2 \in \mathcal{D}(A_2) \). Define \( P_1 : X_2 \to X_1, P_2 : X_1 \to X_2 \) by
\[ (P_1\varphi)(\xi) = r_1(\xi) \int_{\Omega} q_1(\sigma)\varphi(\sigma)d\sigma, \]
\[ (P_2\varphi)(\xi) = r_2(\xi) \int_{\Omega} q_2(\sigma)\varphi(\sigma)d\sigma. \]

Set
\[ X := X_1 \times X_2; \]
\[ A := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad \text{with } \mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2); \]
\[ P := \begin{pmatrix} 0 & P_1 \\ P_2 & 0 \end{pmatrix}, \quad \text{with } \mathcal{D}(P) = X. \]

Then \( \mathcal{R}(P) \subset \mathcal{D}(A) \). Writing \( C = A^{-1} \), we see that \( A \) generates a \( C \)-regularized semigroup \( \{ T(t) \}_{t \geq 0} \) on \( X \) given by
\[ T(t) := \begin{pmatrix} T_1(t)A_1^{-1} & 0 \\ 0 & T_2(t) \end{pmatrix}, \]

and for \( x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(A), 0 \leq s \leq t < 1, \)
\[ C^{-1}PAT(t-s)x := \begin{pmatrix} A_1P_1A_2T_2(t-s)x_2 \\ A_2P_2T_1(t-s)x_1 \end{pmatrix}. \]

It is not hard to see that the operators \( A_1P_1A_2 \) and \( A_2P_2 \) have bounded extensions, and therefore there exists \( M > 0 \) such that
\[ ||C^{-1}PAT(t-s)|| \leq M, \quad 0 \leq s \leq t \leq 1. \]
Put \( \tau = \min\{1, (2M)^{-1}\} \), we get
\[
\left\| \int_0^t \Psi(s)C^{-1}PAT(t-s)xds \right\| \leq \frac{1}{2} \|\Psi\|_{\infty} \|x\|, \quad t \in [0, \tau],
\]
for \( x \in D(A) \), \( \Psi \in C([0, \tau], L_\infty(X)) \), which means (H2) holds. Next, we let \( P_2 = 0 \) for simplicity. Then
\[
(I + P)^{-1} = \begin{pmatrix} I & -P_1 \\ 0 & I \end{pmatrix} \in \mathcal{L}(X).
\]
Therefore, \( 0 \in \rho((I + P)A) \). Set \( C_1 = A^{-1}(I + P)^{-1} \). Then
\[
\mathcal{R}(C_1) = D(A), \quad C^{-1}C_1 = (I + P)^{-1}.
\]
Thus, we see that Theorem 2.1 is applicable to this situation, and yields that
\[
\begin{pmatrix} A_1 & P_1A_2 \\ 0 & A_2 \end{pmatrix}
\]
generates a local \( C_1 \)-regularized semigroup on \( X \).

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