REGULAR ELEMENTS WHICH IS A SUM OF AN IDEMPOTENT AND A LEFT CANCELLABLE ELEMENT

Huanyin Chen and Miaosen Chen

Abstract. Let $M$ be a right $R$-module, and let $a \in \text{End}_RM$ be unit-regular. If $\text{End}_R(\text{Im}a)$ is an exchange ring and $\text{End}_R(\text{Ker}a)$ has stable rank one, it is shown that there exist an idempotent $e \in \text{End}_RM$ and a left cancellable $u \in \text{End}_RM$ such that $a = e + u$ and $aM \cap eM = 0$.

1. INTRODUCTION

A ring $R$ is an exchange ring if for every right $R$-module $A$ and two decompositions $A = M \oplus N = \oplus_{i \in I} A_i$, where $M_R \cong R$ and the index set $I$ is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M \oplus (\oplus_{i \in I} A'_i)$. It is well known that a ring $R$ is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. Clearly, regular rings, $\pi$-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit $C^*$-algebras of real rank zero (cf. [2, Theorem 7.2]) are all exchange rings. We say that a right $R$-module $M$ has the finite exchange property if and only if $\text{End}_RM$ is an exchange ring. A ring $R$ has stable rank one in case $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by$ is a unit of $R$. We know that a right $R$-module $M$ can be cancelled from direct sums if and only if $\text{End}_RM$ has stable rank one. Also we know that every strongly $\pi$-regular ring has stable rank one.

Recall that an element $x \in R$ is clean provided that it is a sum of an idempotent and a unit. We say that a ring $R$ is clean if every element in $R$ is clean. Many author investigated clean rings such as [1],[4-7] and [10-16]. Answering a question of Nilcholson, Camillo and Yu [5, Theorem 5] claimed that every unit-regular ring is clean. But there was a gap in their proof. Camillo and Khurana proved this result

Received April 1, 2004; accepted September 14, 2004.
Communicated by Shun-Jen Cheng.
2000 Mathematics Subject Classification: 15E50, 19B10.
Key words and phrases: Exchange ring, Stable rank one, Idempotent.
by a new route and gave a characterization of unit regular rings. They proved a ring \( R \) is unit-regular if and only if for any \( a \in R \) there exist an idempotent \( e \in R \) and a unit \( u \in R \) such that \( a = e + u \) and \( aR \cap eR = 0 \). In this paper, we extend Camillo and Khurana’s result to exchange rings and get a new characterization of a regular element which is a sum of an idempotent and a left cancellable.

Throughout the paper, every ring is associative with an identity. An element \( x \in R \) is regular if there exists \( h \in R \) such that \( x = xh + hx \). An exchange ring, there exist right \( a \in R \), \( b \in R \) such that \( aM \subseteq Kera \) and \( M \leq b \). Hence, \( Kera \cong \frac{M}{Ima} \), \( X \leq \frac{X}{Y} \). Thus \( X \) can be cancelled from direct sums, so we get a right \( R \)-module isomorphism \( \psi : X_2 \to X_1 \). As \( \text{End}_R(Kera) \) has stable rank one, so has \( \text{End}_R(X_k) \). Thus \( X_1 \) can be cancelled from direct sums, so we get a right \( R \)-module isomorphism \( \psi : X_2 \to Y_1 \).

Theorem 1. Let \( M \) be a right \( R \)-module, and let \( a \in \text{End}_RM \) be unit-regular. If \( \text{End}_R(Ima) \) is an exchange ring and \( \text{End}_R(Kera) \) has stable rank one, then there exist an idempotent \( e \in \text{End}_RM \) and a left cancellable \( u \in \text{End}_RM \) such that \( a = e + u \) and \( aM \cap eM = 0 \).

Proof. Set \( E = \text{End}_RM \). Since \( a \in E \) is regular, we have \( x \in E \) such that \( a = axa \). So \( M = \text{Ima} \oplus (1_M - ax)M = xaM \oplus Kera \). As \( \text{End}_R(Ima) \) is an exchange ring, there exist right \( R \)-modules \( X_1, Y_1 \) such that \( M = \text{Ima} \oplus X_1 \oplus Y_1 \) with \( X_1 \subseteq Kera \) and \( Y_1 \subseteq xaM \). Clearly, \( Kera = \text{Ima} \cap (X_1 \oplus \text{Ima} \oplus Y_1) = X_1 \oplus X_2 \), where \( X_2 = \text{Kera} \cap (\text{Ima} \oplus Y_1) \). Likewise, we have a right \( R \)-module \( Y_2 \) such that \( xaM = Y_1 \oplus Y_2 \). Since \( a \in E \) is unit-regular, we get \( Kera \cong \frac{M}{Ima} \); hence, \( X_1 \oplus X_2 \cong Kera \cong \text{Coker} \cong X_1 \oplus Y_1 \). So we have an isomorphism \( k : X_1 \oplus X_2 \to X_1 \oplus Y_1 \). As \( \text{End}_R(Kera) \) has stable rank one, so has \( \text{End}_R(X_k) \).

Assume that \( (a - hv)(x_1 + y_1 + x_2 + y_2) = 0 \) for any \( x_1 \in X_1 \), \( y_1 \in Y_1 \), \( x_2 \in X_2 \), \( y_2 \in Y_2 \). Then \( a(y_1 + y_2) = x_1 + y_1 + \psi(x_2) \in \text{Ima} \cap (X_1 \oplus Y_1) = 0 \), and then \( x_1 = -y_1 - \psi(x_2) \in X_1 \cap Y_1 = 0 \). It follows from \( a(y_1 + y_2) = 0 \) that \( y_1 + y_2 = (1 - xa)(y_1 + y_2) = Kera \subseteq X_1 \oplus X_2 \cap (Y_1 \oplus Y_2) = 0 \); hence \( y_1 + y_2 = 0 \). This infers that \( y_1 = -y_2 \in Y_1 \cap Y_2 = 0 \), and then \( y_1 = y_2 = 0 \). Furthermore, we get \( \psi(x_2) = -y_1 = 0 \). As \( \psi \) is an isomorphism, we have \( x_2 = 0 \).
Thus \( x_1 + y_1 + x_2 + y_2 = 0 \). This means that \( a - e \in R \) is left cancellable. Let \( u = a - e \). Then \( a = e + u \). Furthermore, we get \( aM \cap eM \subseteq aM \cap (X_1 \oplus Y_1) = 0 \). This implies that \( aM \cap eM = 0 \).

Let \( F \) be a field of characteristic 2. For any \( a \in F[x]/(x^2) \), we have \( b \in F[x]/(x^2) \) such that \( a^2 = ba^3 \). Hence \( F[x]/(x^2) \) is strongly \( \pi \)-regular, and then \( F[x]/(x^2) \) is an exchange ring having stable rank one. In addition, it is easy to show that every left cancellable element in a strongly ring is a unit. It follows by Theorem 1 that for any regular \( c \in F[x]/(x^2) \), there exist an idempotent \( e \in F[x]/(x^2) \) and a unit \( u \in F[x]/(x^2) \) such that \( c = e + u \) and \( c(F[x]/(x^2)) \cap e(F[x]/(x^2)) = 0 \). But we notice that \( F[x]/(x^2) \) is not regular because \( J(F[x]/(x^2)) = (x + (x^2))^2 \neq 0 \). This means that Theorem 1 is a nontrivial generalization of [4, Theorem 1].

**Corollary 2.** Let \( V \) be a right vector space over a division ring, and let \( R = \text{End}_D V \). If \( x \in R \) is congruent modulo \( \text{Soc}(R) \) to a unit, then there exist an idempotent \( e \in R \) and a left invertible \( u \in R \) such that \( a = e + u \) and \( aV \cap eV = 0 \).

**Proof.** Since \( x \in R \) is congruent modulo \( \text{Soc}(R) \) to a unit, by [3, Lemma 3.3], \( \dim_D(\text{Ker} x) = \dim_D(\text{Coker} x) < \infty \). It follows from \( \dim_D(\text{Ker} x) = \dim_D(\text{Coker} x) \) that \( x \in R \) is unit-regular. It follows from \( \dim_D(\text{Ker} x) < \infty \) that \( \text{End}_D(\text{Ker} x) \) has stable rank one. In view of Theorem 1, there exist an idempotent \( e \in R \) and a left cancellable element \( u \in R \) such that \( a = e + u \) and \( aV \cap eV = 0 \). Since \( R \) is a regular ring, we have a \( v \in R \) such that \( u = wu \); hence, \( vu = 1 \). That is, \( u \in R \) is left invertible. Therefore we complete the proof.

Let \( V \) be a right vector space over a division ring, and let \( R = \text{End}_D V \). Very recently, Nicholson et al. proved that for any \( a \in R \), there exist an idempotent \( e \in R \) and an invertible \( u \in R \) such that \( a = e + u \)(see [16, Lemma 1]). But we claim that \( aV \cap eV = 0 \) may be not true. Let \( V \) be an infinitely dimensional vector space over a division ring \( D \) with a basis \( \{x_1, x_2, \cdots, x_n, \cdots\} \). Define \( \sigma : V \to V \) given by \( \sigma(x_i) = x_{i+1} \) (\( i = 1, 2, \cdots \) ) and \( \tau : V \to V \) given by \( \tau(x_1) = 0, \tau(x_i) = x_{i-1} \). Clearly, \( \sigma \tau = 1_V \) and \( \sigma \tau \neq 1_V \). By [16, Lemma 1], there exist an idempotent \( e \in R \) and an invertible \( u \in R \) such that \( \sigma = e + u \). If \( \sigma(V) \cap eV = 0 \), then \( \sigma u^{-1}e = (e + u)u^{-1}e = e u^{-1}e e + e \in aV \cap eV = 0 \); hence, \( \sigma u^{-1}(\sigma - u) = 0 \). This implies that \( \sigma = u \in U(R) \), a contradiction. Therefore \( aV \cap eV \neq 0 \).

Recall that an ideal \( I \) of a ring \( R \) is of bounded index if there is a positive integer \( n \) such that \( x^n = 0 \) for any nilpotent \( x \in I \). Let \( a \in R \). We use \( a_L \) to denote the right \( R \)-module homomorphism from \( R \) to \( R \) given by \( a_L(r) = ar \) for any \( r \in R \).

**Corollary 3.** Let \( I \) be a bounded ideal of an exchange ring \( R \). Then the following hold:
(1) For any unit-regular \( a \in 1 + I \), there exist an idempotent \( e \in R \) and a left cancellable \( u \in R \) such that \( a = e + u \) and \( aR \cap eR = 0 \).

(2) For any unit-regular \( a \in 1 + I \), there exist an idempotent \( e \in R \) and a right cancellable \( u \in R \) such that \( a = e + u \) and \( Ra \cap Re = 0 \).

Proof. (1) Let \( a \in 1 + I \) be unit-regular. Then we have a unit \( x \in 1 + I \) such that \( a = axa \). Hence \( aL \in \text{End}_R R \) is unit-regular. Clearly, \( \text{End}_R(\text{ima}_I) \) is an exchange ring. On the other hand, \( \text{End}_R(Kera_L) = (1 - xa)R(1 - xa) \). Since \( I \) is a bounded ideal of \( R \), \( (1 - xa)R(1 - xa) \) is an exchange ring of bounded index. By [18, Corollary 4], \( \text{End}_R(Kera_L) \) has stable rank one. It follows by Theorem 1 that there exist an idempotent \( eL \in \text{End}_R R \) and a left cancellable \( uL \in \text{End}_R R \) such that \( aL = eL + uL \) and \( aL \cap eL = 0 \). Let \( e = eL(1) \) and \( u = uL(1) \). Then \( e \in R \) is an idempotent and \( u \in R \) is left cancellable, as required.

(2) Let \( R^{op} \) be the opposite ring of \( R \). Then \( I^{op} \) is a bounded ideal of the exchange ring \( R^{op} \). Applying (1) to \( a^{op} \in R^{op} \), we obtain the result.

Let \( I \) be an ideal of a ring \( R \). We say that \( I \) has stable rank one provided that \( aR + bR = R \) with \( a \in 1 + I \) and \( b \in R \) implies that there exists \( y \in R \) such that \( a + by \) is a unit of \( R \). An ideal \( I \) of an exchange ring \( R \) has stable rank one if and only if for any regular \( a \in 1 + I \), there exists a unit \( u \in R \) such that \( a = auu \) (See [7, Proposition 2.3]). It is well known that every bounded ideal of a regular ring has stable rank one. We note that an ideal \( I \) has stable rank one only depends on the ring structure of \( I \) and doesn’t depend on the choice of \( R \). In other words, \( I \) has stable rank one as an ideal of \( R \) if and only if \( I \) has stable rank one as a non-unital ring.

Theorem 4. Let \( I \) be an ideal of an exchange ring \( R \). If \( I \) has stable rank one, then for any regular \( a \in 1 + I \), there exist an idempotent \( e \in I \) and a left cancellable \( u \in 1 + I \) such that \( a = e + u \) and \( aR \cap eR = 0 \).

Proof. Let \( a \in 1 + I \) be regular. Then \( a = axa \) for some \( x \in R \). Since \( I \) has stable rank one, it follows by [7, Proposition 2.3] that \( a \in R \) is unit-regular. This means that \( aL \) is unit-regular. Obviously, \( \text{End}_R(\text{ima}_I) \) is an exchange ring and \( \text{End}_R(Kera_L) \) has stable rank one. Similarly to Theorem 1, we get \( R = aL \oplus (1 - axa)R = xL \oplus (1 - axa)R = xL \oplus (1 - xL)R \). Since \( R \) is an exchange ring, we have right \( R \)-modules \( X_1, Y_1 \) such that \( R = aR \oplus X_1 \oplus Y_1 \) with \( X_1 \subset (1 - xa)R \) and \( Y_1 \subset xaR \). Furthermore, we have right \( R \)-modules \( X_2, Y_2 \) such that \( R = xL \oplus (1 - xL)R = X_2 \oplus Y_2, \) where \( k : X_1 \oplus Y_1 \cong X_2 \oplus Y_2 \) and \( \psi : X_2 \cong Y_1 \). Let \( h : R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \rightarrow X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = R \) given by \( h(x_1 + x_2 + y_1 + y_2) = k^{-1}(x_1 + x_2) + y_1 \) for any \( x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2 \). Let \( v : R = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \rightarrow R \) given by \( v(x_1 + x_2 + y_1 + y_2) = x_1 + x_2 + y_1 \) for any \( x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2 \).
Let $I$ be an ideal of a cohopfian exchange ring $R$. Since $R$ is an exchange ring, so is the opposite ring $R^{op}$. Also we know that if $I$ has stable rank one then so does $I^{op}$. Applying Theorem 4 to the ideal $I^{op}$ of the ring $R^{op}$, we prove that for any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a right cancellable $u \in 1 + I$ such that $a = e + u$ and $Ra \cap Re = 0$. We note that the matrix $[i_{i,j}] \in CF_{MN}(R)$ is left cancellable, while it is not right cancellable. We don’t know whether ”a left cancellable $u \in 1 + I$” could be replaced by ”a unit $u \in 1 + I$ in the proceeding theorem. A ring $R$ is cohopfian if any injective right $R$-module homomorphism from $R$ to $R$ is an isomorphism. As a consequence of Theorem 4, we now derive the following.

**Corollary 5.** Let $I$ be an ideal of a cohopfian exchange ring $R$. Then the following are equivalent:

1. $I$ has stable rank one.
2. For any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$.

**Proof.** (1) $\Rightarrow$ (2) Let $a \in 1 + I$ be regular. By Theorem 4, there exist an idempotent $e \in I$ and a left cancellable $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$. Let $u_L : R \rightarrow R$ given by $u_L(r) = ur$ for any $r \in R$. Since $u \in R$ is cancellable, $u_L$ is injective. As $R$ is a cohopfian ring, $u_L$ is an isomorphism. Assume that $u_Lv = 1 = vu_L$ for a $v \in End_RR$. This infers that $u = v(1)^{-1} \in U(R)$, as required. (2) $\Rightarrow$ (1) For any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$. Hence $au^{-1}e = (e + u)u^{-1}e = eu^{-1}e + e \in aR \cap eR = 0$, and then $au^{-1}(a - u) = 0$. This gives $a = au^{-1}a$. So $I$ has stable rank one by [7, Proposition 2.3].

Recall that a ring $R$ is said to be strongly $\pi$-regular in case for any $x \in R$ there exist a positive integer $n$ and a $y \in R$ such that $x^n = x^{n+1}y$. A right $R$-module $M$ is said to satisfy Fitting’s lemma if, for all $f \in End_RM$, there exists
a positive integer \( n \) such that \( M = f^n(M) \oplus \text{Ker}(f^n) \). It is well known that a module satisfies Fitting’s lemma if and only if its endomorphism ring is a strongly \( \pi \)-regular ring. Also we know that every strongly \( \pi \)-regular ring is a cohopfian exchange ring having stable rank one. Let \( R \) be a strongly \( \pi \)-regular ring. Using Corollary 5, we prove that \( x \in R \) is regular if and only if there exist an idempotent \( e \in R \) and a unit \( u \in R \) such that \( a = e + u \) and \( aR \cap eR = 0 \).

Let \( R = M_2(F[x]/(x^2)) \), where \( F \) is a field. Then \( R \) is strongly \( \pi \)-regular, so it is a clean ring. Let \( a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R \), and let \( u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then \( a = auu \) with \( u \in U(R) \); hence, \( a \) is unit-regular. Thus we have an idempotent \( e \in R \) and a unit \( u \in R \) such that \( a = e + u \) and \( aR \cap eR = 0 \). But \( a^2 \) can not be written in the form above. This is because \( a^2 \) is not regular. In other words, some elements in a ring \( R \) can be written in this form, while the other elements can not be written in this form.

A ring \( R \) is a \( \pi \)-regular ring in case for any \( a \in R \) there exists a positive integer \( n(x) \) such that \( a^{n(x)} = a^{n(x)}ca^{n(x)} \) for a \( c \in R \). Clearly, every \( \pi \)-regular ring is an exchange ring.

**Corollary 6.** Let \( I \) be an ideal of a \( \pi \)-regular ring \( R \). Then the following are equivalent:

1. \( I \) has stable rank one.
2. For any regular \( a \in 1 + I \), there exist an idempotent \( e \in I \) and a unit \( u \in 1 + I \) such that \( a = e + u \) and \( aR \cap eR = 0 \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( a \in 1 + I \) be regular. By Theorem 4, there exist an idempotent \( e \in I \) and a left cancellable \( u \in 1 + I \) such that \( a = e + u \) and \( aR \cap eR = 0 \). Since \( R \) is \( \pi \)-regular ring, we have a positive integer \( n \) such that \( u^n = u^n vu^n \) for a \( v \in R \). Hence \( u^n (1 - vu^n) = 0 \). As \( u \) is left cancellable, we deduce that \( vu^n = 1 \). Clearly, \( v \in 1 + I \). From \( vu^n + 0 = 1 \), we can find a \( y \in R \) such that \( v = v + 0 \times y \in U(R) \) because \( I \) has stable rank one. This means that \( u \in U(R) \).

(2) \( \Rightarrow \) (1) is analogous to Corollary 5.

Let \( I \) be an ideal of a \( \pi \)-regular ring \( R \). Analogously, we prove that \( I \) has stable rank one if and only if for any regular \( a \in 1 + I \), there exist an idempotent \( e \in I \) and a unit \( u \in 1 + I \) such that \( a = u - e \) and \( aR \cap eR = 0 \). Let \( R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0 \text{ and } 3 \nmid b \text{ and } 5 \nmid b\} \). By [1, Proposition 16], each element \( a \in R \) can be written in the form \( a = u + e \) or \( a = u - e \) where \( u \in U(R) \) and \( e \in R \) is an idempotent. But \( R \) is not a clean ring. In other words, there exists an element \( a \in R \) which is not a sum of an idempotent and a unit can be written in the form \( a = u - e \) where \( u \in U(R) \) and \( e \in R \) is an idempotent.
**Corollary 7.** Let $R$ be a regular ring, and let $a \in R$. If $RaR$ has stable rank one, then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $(1 - a)R \cap (1 - e)R = 0$.

**Proof.** Let $I = RaR$ and $b = 1 - a$. Then $I$ has stable rank one and $b \in 1 + I$. By Theorem 4, there exist an idempotent $f \in I$ and a left cancellable $v \in 1 + I$ such that $b = f + v$ and $bR \cap fR = 0$. As $R$ is regular, there exists a $w \in R$ such that $v = vwv$. So we see that $v \in 1 + I$ is left invertible. On the other hand, $I$ has stable rank one. Hence $v \in 1 + I$ is a unit. Let $e = 1 - f$. Then $e \in R$ is an idempotent. In addition, we have $a = 1 - b = e + (-u)$. Set $u = -v$. Then $v \in R$ is a unit and $a = e + u$. Furthermore, we have $(1 - a)R \cap (1 - e)R = 0$, as required. ■

Let $R$ be a regular ring, and let $A = (a_{ij}) \in M_n(R)$. If every $Ra_{ij}R$ has stable rank one, we claim that there exist an idempotent $E \in M_n(R)$ and an invertible $U \in M_n(R)$ such that $A = E + U$ and $(I_n - A)M_n(R) \cap (I_n - E)M_n(R) = 0$. Set $I = \sum_{1 \leq i,j \leq n} Ra_{ij}R$. One easily checks that $I$ has stable rank one. Clearly, $M_n(R)$ is regular. It follows from $M_n(R)AM_n(R) \subseteq M_n(I)$ that $M_n(R)AM_n(R)$ has stable rank one. In view of Corollary 7, we are done.

Let $LT M_n(R)(U TM_n(R))$ be the ring of all lower(upper) triangular matrices over a ring $R$. We note that $LT M_2(R)$ is not a regular ring even if $R$ is regular. The reason is that $egin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not a regular element in $LT M_2(R)$. Now we investigate the conditions under which a triangular matrix can be written in the form above.

**Theorem 8.** Let $R$ be regular, and let $A = (a_{ij}) \in LT M_n(R)$. If every $Ra_{ii}R$ has stable rank one, then there exist an idempotent $E \in LT M_n(R)$ and an invertible $U \in LT M_n(R)$ such that $A = E + U$ and $(I_n - A)LT M_n(R) \cap (I_n - E)LT M_n(R) = 0$.

**Proof.** If $n = 1$, then the result follows by Corollary 7. Assume that the result holds for $n = k(k \geq 1)$. Let $n = k + 1$. Given any $A = \begin{pmatrix} A_1 & 0 \\ * & a_{nn} \end{pmatrix}$ with any $Ra_{ii}R$ has stable rank one, by the hypothesis, we can find an idempotent $E_1 \in LT M_k(R)$ and an invertible $U_1 \in LT M_k(R)$ such that $A = E_1 + U_1$ and $(I_k - A_1)LT M_k(R) \cap (I_k - E_1)LT M_k(R) = 0$. Similarly, we can find an idempotent $e_2 \in R$ and an invertible $u_2 \in R$ such that $a_{nn} = e_2 + u_2$ and $(1 - a_{nn})R \cap (1 - e)R = 0$. One easily checks that $A = \text{diag}(E_1, e_2) + \begin{pmatrix} U_1 & 0 \\ * & u_2 \end{pmatrix}$. Clearly, $\text{diag}(E_1, e_2) \in M_n(R)$ is an idempotent matrix and $\begin{pmatrix} U_1 & 0 \\ * & u_2 \end{pmatrix} \in M_n(R)$ is an invertible triangular matrix. Furthermore, we verify that $(I_n - A)LT M_n(R) \cap (I_n - E)LT M_n(R) = 0$. By induction, we complete the proof. ■
Corollary 9. Let $R$ be unit-regular, and let $A \in LTM_n(R)$. Then there exist an idempotent $E \in LTM_n(R)$ and an invertible $U \in LTM_n(R)$ such that $A = E + U$ and $(I_n - A)LTM_n(R) \cap (I_n - E)LTM_n(R) = 0$.

Proof. Since $R$ is unit-regular, it is shown that every $Ra_{ii}R$ has stable rank one. Therefore the result follows by Theorem 8.

Let $R$ be unit-regular, and let $A \in UTM_n(R)$. Analogously, we deduce that there exist an idempotent $E = (e_{ij}) \in UTM_n(R)$ and an invertible $U = (u_{ij}) \in UTM_n(R)$ such that $A = E + U$ and $(I_n - A)UTM_n(R) \cap (I_n - E)UTM_n(R) = 0$. Define $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ | \ a + c = b + d, a, b, c, d \in R \right\}$.

Corollary 10. Let $A = (a_{ij})$ be a $2 \times 2$ matrix over a unit-regular ring $R$. If $a_{11} + a_{21} = a_{12} + a_{22}$, then there exist an idempotent $E = (e_{ij}) \in M_2(R)$ and an invertible $U = (u_{ij}) \in M_2(R)$ such that

1. $A = E + U$.
2. $e_{11} + e_{21} = e_{12} + e_{22}$.
3. $u_{11} + u_{21} = u_{12} + u_{22}$.

Proof. Construct a map $\psi : QM_2(R) \to TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a + c & 0 \\ c & d - c \end{pmatrix}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in TM_2(R)$, we have $\psi \left( \begin{pmatrix} x - z & x - y - z \\ z & y + z \end{pmatrix} \right) = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$. Thus $\psi$ is an epimorphism. It is easy to verify that $\psi$ is a monomorphism; hence, it is a ring isomorphism. Therefore we complete the proof by Theorem 8.

Let $A = (a_{ij})$ be a $2 \times 2$ matrix over a unit-regular ring $R$. If $a_{11} + a_{12} = a_{21} + a_{22}$, analogously to the consideration above, we conclude that there exist an idempotent $E = (e_{ij}) \in M_2(R)$ and an invertible $U = (u_{ij}) \in M_2(R)$ such that

1. $A = E + U$; (2) $e_{11} + e_{12} = e_{21} + e_{22}$; (3) $u_{11} + u_{12} = u_{21} + u_{22}$.

ACKNOWLEDGEMENT

The author would like to thank the referee for his/her helpful comments and suggestions, which lead to the new version of this paper.
REFERENCES


Huanyin Chen\textsuperscript{1} and Miaosen Chen\textsuperscript{2}
Department of Mathematics,
Zhejiang Normal University,
Jinhua 321004, People’s Republic of China
\textsuperscript{1}E-mail: chyzxl@hunnu.edu.cn
\textsuperscript{2}E-mail: miaosen@mail.jhptt.zj.cn