TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 2, pp. 487-500, April 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

APPROXIMATION OF COMMON FIXED POINTS
OF FAMILIES OF NONEXPANSIVE MAPPINGS*

L. C. Ceng1, P. Cubiotti2 and J. C. Yao3,*

Abstract. Let $X$ be a reflexive and smooth Banach space which has a weakly sequentially continuous duality mapping. We consider in this paper the iteration scheme $x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1})T_{n+1}x_n$ for infinitely many nonexpansive maps $T_1, T_2, T_3, \ldots$ in $X$ as well as for finitely many nonexpansive retraction. We establish several strong convergence results which generalize [10, Theorem 3.3] and [10, Theorem 4.1] from Hilbert space setting to Banach space setting.

1. INTRODUCTION

In 1967 for $N$ nonexpansive maps $T_1, T_2, \ldots, T_N$, Halpern [7] first introduced the iteration scheme

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1})T_{n+1}x_n$$

in which he considered the case when $y = 0$ and $N = 1$; i.e., one map $T$. He proved that the conditions $\lim_{n \to \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ were necessary conditions for the convergence of the iterates to a fixed point of $T$. In 1977 Lions [9] considered the above scheme with the additional assumption $\lim_{n \to \infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$ on the parameters and proved convergence of the iterates. In 1983 Reich [13] posed the following problem:

Received January 7, 2006, accepted March 7, 2006.
Communicated by Wen-Wei Lin.
2000 Mathematics Subject Classification: 47H09, 65J15.
Key words and phrases: Common fixed point, Sunny and nonexpansive retraction, Nonexpansive mapping, Banach space.
* The authors thank the referee for his(her) valuable comments and suggestions that improved the original manuscript greatly.
1 This research was partially supported by the National Science Foundation of China (10771141) and Shanghai Leading Academic Descipline Project (T0401).
3 This research was partially supported by a grant from the National Science Council.
+ Corresponding author
In a Banach space, what conditions on the sequence \( \{\lambda_n\} \) of parameters will ensure convergence of the iterates?

In 1992 Wittmann [19] proved convergence of the iterates in a Hilbert space under the assumption that the parameters satisfy \( \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty \) in addition to the above two necessary conditions. In 1994 under the assumption that the parameters satisfy the two necessary conditions and are decreasing, Reich [12] proved strong convergence of the iterates for the case of a single map (i.e., \( N = 1 \)) in a uniformly smooth Banach space which has a weakly continuous duality map.

In 1996 Bauschke [1] generalized Wittmann’s result to finitely many maps where \( T_n := T_{n \mod N} \). The additional condition imposed by him on the parameters was \( \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty \). He also provided an algorithmic proof which has been used successfully with modifications by many authors [4, 15, 23]. In 1997 Shioji and Takahashi [17] extended Wittmann’s result to a Banach space. This paper provides some answers to the problem posed by Reich [13] by introducing a new condition on the parameters \( \lim_{n \to \infty} \lambda_n / \lambda_{n+N} = 1 \) in the framework of a Hilbert space. Shimizu and Takahashi (see [15, Theorem 1]) in 1997 considered the above iteration scheme with the necessary conditions on the parameters and some additional conditions imposed on the mappings. In 2003 O’Hara, Pillay and Xu [10] established the following strong convergence result in a Hilbert space which generalizes Theorem 1 of Shimizu and Takahashi [15].

**Theorem 1.1.** [10, Theorem 3.3]. Let \( \{\lambda_n\} \subset (0, 1) \) satisfy \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=1}^{\infty} \lambda_n = \infty \). Let \( C \) be a nonempty, closed and convex subset of a Hilbert space \( H \) and let \( T_n : C \to C (n = 1, 2, 3, \ldots) \) be nonexpansive mappings such that

\[
F := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset
\]

where \( \text{Fix}(T_n) = \{x \in C : x = T_n x\}, \ n = 1, 2, 3, \ldots \). Assume that \( V_1, \ldots, V_N : C \to C \) are nonexpansive mappings with the property: for all \( k = 1, 2, \ldots, N \) and for any bounded subset \( \tilde{C} \) of \( C \), there holds

\[
\lim_{n \to \infty} \sup_{x \in \tilde{C}} \|T_n x - V_k(T_n x)\| = 0.
\]

For \( x_0 \in C \) and \( y \in C \) define

\[
x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \geq 0.
\]

Then \( \{x_n\} \) converges strongly to \( Py \) where \( P \) is the projection from \( H \) onto \( \bigcap_{k=1}^{N} \text{Fix}(V_k) \).
Furthermore for the same iteration scheme (1) with finite many maps $T_1, T_2, ..., T_N$, O’Hara, Pillay and Xu [10] established the following complementary result to Theorem 3.1 of Bauschke [1] with condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ replaced by condition $\lim_{n \to \infty} \lambda_n / \lambda_{n+N} = 1$.

**Theorem 1.2.** [10, Theorem 4.1]. Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$ and let $T_1, T_2, ..., T_N$ be nonexpansive self-mappings of $C$ with $F := \bigcap_{i=1}^{N} \text{Fix}(T_i) \neq \emptyset$. Assume that

$$F = \text{Fix}(T_N...T_1) = \text{Fix}(T_1T_N...T_2) = \cdots = \text{Fix}(T_{N-1}T_{N-2}...T_1T_N).$$

Let $\{\lambda_n\} \subset (0, 1)$ satisfy the following conditions:

1. $\lim_{n \to \infty} \lambda_n = 0$;
2. $\sum_{n=1}^{\infty} \lambda_n = \infty$;
3. $\lim_{n \to \infty} \lambda_n / \lambda_{n+N} = 1$.

Given points $x_0, y \in C$, the sequence $\{x_n\} \subset C$ is defined by

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to $P_F y$ where $P_F$ is the projection of $C$ onto $F$.

Let $X$ be a reflexive Banach space which has a weakly sequentially continuous duality map. For example, every $l^p$ ($1 < p < \infty$) space has a weakly sequentially continuous duality map with gauge function $\varphi(t) = t^{p-1}$. In this paper the iteration scheme (1) is considered for infinitely many nonexpansive maps $T_1, T_2, T_3, ...$ in $X$. Theorem 3.3 of O’Hara, Pillay and Xu [10] is extended to the setting of Banach space $X$ and it is shown that the sequence of iterates converges strongly to $P y$ where $P$ is some sunny and nonexpansive retraction. For this same iteration scheme (1) with finitely many nonexpansive maps $T_1, T_2, ..., T_N$ in $X$, Theorem 4.1 of O’Hara, Pillay and Xu [10] is also extended to the setting of Banach space $X$ under the same conditions imposed by them on the sequence $\{\lambda_n\}$ of parameters. The iterates converge strongly to $P y$ where $P$ is the sunny and nonexpansive retraction onto the intersection of the fixed point sets of the $T_i$, $i = 1, 2, ..., N$.

2. Preliminaries

Throughout this paper let $X$ be a real Banach space and $X^*$ be its dual space. Let $C$ be a nonempty subset of $X$ and $T : C \to C$ be a mapping of $C$ into itself. $T$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The fixed point set of $T$ is denoted by $\text{Fix}(T) := \{x \in C : Tx = x\}$. The notation $\rightharpoonup$ denotes weak convergence and the notation $\to$ denotes strong convergence. By a gauge function
we mean a continuous strictly increasing function $\varphi$ defined on $R^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r \to \infty} \varphi(r) = \infty$. The mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{ x^* \in X^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \varphi(\|x\|) \}, \quad \forall x \in X$$

is called the duality mapping with gauge function $\varphi$. In particular the duality mapping with gauge function $\varphi(t) = t$ denoted by $J$ is referred to as the normalized duality mapping. Browder [2] initiated the study of certain classes of nonlinear operators by means of the duality mapping $J_\varphi$. Set for every $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r) dr.$$ 

Then it is known [8, p. 1350] that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|x\|) = \|x\|^2/2$, that is

$$\text{(2) } J(x) = \partial \Phi(\|x\|) = \{ f \in X^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y-x, f \rangle \ \forall y \in X \} \quad \forall x \in X.$$ 

We will use the following properties of duality mappings.

**Proposition 2.1.** [22, p. 193-194].

(i) $J = I$ (i.e., the identity mapping of $X$) if and only if $X$ is a Hilbert space.

(ii) $J$ is surjective if and only if $X$ is reflexive.

(iii) $J_\varphi(\lambda x) = \text{sign}(\lambda)(\varphi(\|\lambda \cdot x\|)/\|x\|)J(x)$ $\forall x \in X \setminus \{0\}, \lambda \in R$ where $R$ is the set of all real numbers; in particular $J(-x) = -J(x), \forall x \in X$.

Recall that a Banach space $X$ is said to satisfy Opial’s condition [11] if for any sequence $\{x_n\}$ in $X$ the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in X, y \neq x$. It is known [18] that any separable Banach space can be equivalently normed so that it satisfies Opial’s condition. Recall also that $X$ is said to have a weakly sequentially continuous duality mapping if there exists a gauge function $\varphi$ such that the duality mapping $J_\varphi$ is single-valued and continuous from the weak topology to the weak* topology; i.e., for any sequence $\{x_n\}$ in $X$, if $x_n \rightharpoonup x$ in $X$, then $J_\varphi(x_n) \rightharpoonup J_\varphi(x)$ in the weak* topology of $X$. A space with a weakly sequentially continuous duality mapping is easily seen to satisfy Opial’s condition; see [2] for more details. Every $l^p$ space $(1 < p < \infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$. 


Let $U = \{x \in X : \|x\| = 1\}$, the unit sphere of $X$. The norm of $X$ is said to be Gâteaux differentiable if the limit

$$
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
$$

exists for each $x, y \in U$. In this case $X$ is said to be smooth. It is known [24] that $X$ is smooth if and only if the normalized duality mapping $J$ is single-valued. In this case, the normalized duality mapping $J$ is continuous from the strong topology to the weak$^*$ topology. Moreover, if $X$ admits a weakly sequentially continuous duality mapping, then $X$ satisfies Opial’s condition and $X$ is smooth, see Lemma 1 in [25].

In the sequel we will use the following concepts and lemmas.

**Lemma 2.1.** (see [6, Lemma 4]). Let $X$ be a Banach space satisfying Opial’s condition and let $C$ be a nonempty, closed and convex subset of $X$. Let $T : C \to C$ be a nonexpansive mapping. Then $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in $C$ which converges weakly to $x$ and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.

**Lemma 2.2.** Let $\varphi$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$, and let

$$
\Phi(t) = \int_0^t \varphi(r)dr.
$$

Then there holds the following inequality

$$
\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y) \rangle, \forall x, y \in X,
$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$.

**Proof.** The proof of this lemma is essentially due to Lim and Xu [8]. For the completeness, we give its proof. Indeed it is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at $x$, that is,

$$
J_\varphi(x) = \partial \Phi(\|x\|) = \{f \in X^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, f \rangle \forall y \in X\}.
$$

Consequently, it follows that for each $x, y \in X$

$$
\Phi(\|x\|) - \Phi(\|x + y\|) \geq \langle x - (x + y), j_\varphi(x + y) \rangle \forall j_\varphi(x + y) \in J_\varphi(x + y).
$$

The conclusion follows from the above inequality. ■
Let $C$ be a convex subset of $X$, $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$, i.e., $Px = x$ for each $x \in K$. $P$ is said to be sunny if $P(Px + t(x - Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px + t(x - Px) \in C$. If there is a sunny and nonexpansive retraction from $C$ onto $K$, $K$ is said to be a sunny and nonexpansive retract of $C$. For a sunny and nonexpansive retraction, there exists the following useful characterization.

**Lemma 2.3** [16, Proposition 4, p. 59]. Let $C$ be a convex subset of a smooth Banach space $X$, $K$ be a nonempty subset of $C$ and let $P$ be a retraction from $C$ onto $K$. Then $P$ is sunny and nonexpansive if and only if for all $x \in C$ and $y \in K$,

$$\langle x - Px, J(y - Px) \rangle \leq 0.$$ 

Hence there is at most one sunny and nonexpansive retraction from $C$ onto $K$. More information regarding sunny and nonexpansive retractions can be found in [5, 14].

**Remark 2.1.** If $X = H$ is a real Hilbert space and $C$ is a nonempty, closed and convex subset of $H$, then every nearest point projection of $H$ onto $C$ is a sunny and nonexpansive retraction of $H$ onto $C$ where the mapping $P_C : H \rightarrow C$ is defined as follows: for each $x \in H$, $P_Cx$ is the unique element of $C$ that satisfies

$$\|x - P_Cx\| = d(x, C) := \inf_{y \in C} \|x - y\|.$$ 

Indeed it is easy to see that $P_C$ is a retraction of $H$ onto $C$. Moreover it follows from Lemma 2.3 in [10] that for all $x \in H$ and $y \in C$,

$$\langle x - P_Cx, P_Cx - y \rangle \geq 0.$$ 

According to Lemma 2.3, we know that $P_C$ is a sunny and nonexpansive retraction of $H$ onto $C$.

**Lemma 2.4.** (see [1]). Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ such that $\lim_{n \to \infty} \lambda_n = 0$. Then

$$\sum_{n=1}^{\infty} \lambda_n = \infty \iff \prod_{n=1}^{\infty} (1 - \lambda_n) = 0.$$

**Lemma 2.5.** [10, Lemma 2.2]. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ that satisfies $\lim_{n \to \infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any one of the following conditions:

(a) For all $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\varepsilon.$$

(b) $a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_nc_n$ where $\limsup_{n \to \infty} c_n \leq 0$.

Then $\lim_{n \to \infty} a_n = 0$. 
The proof of Lemma 2.5 can be found in [20].

3. STRONG CONVERGENCE BY IMPOSING CONDITIONS ON THE MAPPINGS

In this section we establish the following strong convergence result in a real Banach space which generalizes Theorem 3.3 of O’Hara, Pillay and Xu [10].

**Theorem 3.1.** Let \( \{ \lambda_n \} \subset (0, 1) \) satisfy \( \lim_{n \to \infty} \lambda_n = 0 \) and \( \sum_{n=1}^{\infty} \lambda_n = \infty \). Let \( X \) be a reflexive Banach space which has a weakly sequentially continuous duality mapping \( J_\varphi \) with gauge function \( \varphi \). Let \( C \) be a nonempty, closed and convex subset of \( X \) and let \( T_n : C \to C (n = 1, 2, 3, \ldots) \) be nonexpansive mappings such that

\[
F := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset.
\]

Assume that \( V_1, \ldots, V_N : C \to C \) are nonexpansive mappings with the following property: for all \( k = 1, 2, \ldots, N \) and for any bounded subsets \( C \) of \( C \), there holds

\[
\lim_{n \to \infty} \sup_{x \in C} \| T_n x - V_k(T_n x) \| = 0.
\]

For \( x_0 \in C \) and \( y \in C \) define

\[
x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \quad n \geq 0.
\]

If there exists a sunny and nonexpansive retraction \( P \) of \( C \) onto \( \bigcap_{k=1}^{N} \text{Fix}(V_k) \), then

\[
\lim_{n \to \infty} \sup_{x \in C} (y - P y, J_\varphi(x_n - P y)) \leq 0.
\]

Suppose additionally that \( Py \) lies in \( F \). Then \( x_n \to Py \).

**Proof.** The proof given below employs the same idea as in the proof of Theorem 3.3 [10]. We note that assumption (3) implies that \( \bigcap_{k=1}^{N} \text{Fix}(V_k) \supseteq F \). We proceed with the following steps.

**Step 1.** We claim that for all \( n \geq 0 \),

\[
\| x_n - f \| \leq \max\{\|x_0 - f\|, \|y - f\|\} \quad \forall f \in F.
\]

Indeed we use an inductive argument. The result is clearly true for \( n = 0 \). Suppose the result is true for \( n \). Let \( f \in F \). Then by the nonexpansivity of \( T_{n+1} \),

\[
\| x_{n+1} - f \| = \| \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n - f \|
\]

\[
= \| \lambda_{n+1}(y - f) + (1 - \lambda_{n+1})(T_{n+1} x_n - f) \|
\]

\[
\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\| T_{n+1} x_n - f \|
\]

\[
\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\| x_n - f \|
\]
\[
\leq \lambda_{n+1} \max\{\|x_0 - f\|, \|y - f\|\} \\
+ (1 - \lambda_{n+1}) \max\{\|x_0 - f\|, \|y - f\|\} \\
= \max\{\|x_0 - f\|, \|y - f\|\}.
\]

**Step 2.** We claim that \(\{x_n\}\) is bounded. Indeed for all \(n \geq 0\) and for any \(f \in F\),
\[
\|x_n\| \leq \|x_n - f\| + \|f\| \\
\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\]

**Step 3.** We claim that \(\{T_{n+1}x_n\}\) is bounded. Indeed for all \(n \geq 0\) and for any \(f \in F\),
\[
\|T_{n+1}x_n\| \leq \|T_{n+1}x_n - f\| + \|f\| \\
\leq \|x_n - f\| + \|f\| \\
\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\]

**Step 4.** We claim that \(x_{n+1} - T_{n+1}x_n \to 0\). Indeed we have
\[
\|x_{n+1} - T_{n+1}x_n\| = \lambda_{n+1}\|y - T_{n+1}x_n\| \\
\leq \lambda_{n+1}(\|y\| + \|T_{n+1}x_n\|) \\
\leq \lambda_{n+1}(\|y\| + M) \quad \text{for some } M.
\]
Since \(\lambda_{n+1} \to 0\), we obtain \(x_{n+1} - T_{n+1}x_n \to 0\).

**Step 5.** We claim that \(\limsup_{n \to \infty} \langle y - Py, J_\varphi(x_{n+1} - Py)\rangle \leq 0\). Indeed, since \(X\) is reflexive and \(\{x_n\}\) is bounded by Step 2, there exists a subsequence \(\{x_{n_j+1}\}\) of \(\{x_n\}\) such that \(x_{n_j+1} \to p\) for some \(p \in C\) and
\[
\limsup_{n \to \infty} \langle y - Py, J_\varphi(x_{n+1} - Py)\rangle = \lim_{j \to \infty} \langle y - Py, J_\varphi(x_{n_j+1} - Py)\rangle.
\]
By our assumption we have for any \(k = 1, 2, ..., N\) and for \(\hat{C} = \{x_n\}\),
\[
0 = \limsup_{n \to \infty} \|T_{n+1}x - V_k(T_{n+1}x)\| \geq \limsup_{n \to \infty} \|T_{n+1}x_n - V_k(T_{n+1}x_n)\| \\
\geq \limsup_{j \to \infty} \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\|.
\]
Thus
\[\lim_{j \to \infty} \| T_{n,j+1} x_{n_j} - V_k(T_{n,j+1} x_{n_j}) \| = 0 \quad \text{for all } k = 1, 2, \ldots, N.\]

Therefore \( p \in Fix(V_k) \) for \( k = 1, 2, \ldots, N \) by Lemma 2.1; i.e., \( p \in \bigcap_{k=1}^{N} Fix(V_k) \). Thus we deduce from Lemma 2.3 that
\[
\limsup_{n \to \infty} \langle y - Py, J_\varphi(x_{n+1} - Py) \rangle = \lim_{j \to \infty} \langle y - Py, J_\varphi(x_{n,j+1} - Py) \rangle \leq 0,
\]
since \( p \in \bigcap_{k=1}^{N} Fix(V_k) \).

**Step 6.** Suppose additionally that \( Py \) lies in \( F \). Then we claim that \( x_n \to Py \). Indeed using Lemma 2.2, we obtain
\[
\Phi(\|x_{n+1} - Py\|) = \Phi(\|(1 - \lambda_{n+1})(T_{n+1}x_n - Py) + \lambda_{n+1}(y - Py)\|) \\
\leq \Phi(\|(1 - \lambda_{n+1})(T_{n+1}x_n - Py)\|) + \lambda_{n+1}(y - Py, J_\varphi(x_{n+1} - Py)) \\
\leq (1 - \lambda_{n+1})\Phi(\|x_n - Py\|) + \lambda_{n+1}(y - Py, J_\varphi(x_{n+1} - Py)).
\]
Applying Lemma 2.5, we conclude that \( \Phi(\|x_{n+1} - Py\|) \to 0 \); that is, \( \|x_{n+1} - Py\| \to 0 \). Consequently, \( x_n \to Py \). The proof is now complete.

4. **Strong Convergence by Imposing Conditions on the Parameters**

In 1996 Bauschke [1] defined the following control conditions on the parameters \( \{\lambda_n\} \):

\[\text{[B1]} \lim_{n \to \infty} \lambda_n = 0.\]
\[\text{[B2]} \sum_{n=1}^{\infty} \lambda_n = \infty.\]
\[\text{[B3]} \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.\]

In 2003 O’Hara, Pillay and Xu [10] replaced [B3] by the condition
\[\text{[N3]} \lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+1}} = 1.\]
This condition also improves Lions’ condition [9] as follows
\[\text{[L3]} \lim_{n \to \infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}} = 0.\]

Note that both [N3] and [B3] cover the natural candidate of \( \lambda_n = (n+1)^{-1} \) but [L3] does not. However [B3] and [N3] are independent of each other (even coupled with conditions [B1] and [B2])); see [20]. Theorem 4.1 given in [10] is a complementary
result to Theorem 3.1 of Bauschke [1] with condition [B3] replaced by condition [N3]. Its proof employs the same idea as in the proof of Theorem 3.1 [1]. We will now extend Theorem 4.1 [10] to the setting of Banach space $X$ under the same conditions as those imposed on the parameters $\{\lambda_n\}$ in [10, Theorem 4.1].

We consider $N$ maps $T_1, T_2, \ldots, T_N$. For $n > N$, set

$$T_n := T_{n \mod N},$$

where $n \mod N$ is defined as follows: if $n = kN + l$ with $0 \leq l < N$, then

$$n \mod N := \begin{cases} l, & \text{if } l \neq 0, \\ N, & \text{if } l = 0. \end{cases}$$

**Theorem 4.1.** Let $X$ be a reflexive Banach space which has a weakly sequentially continuous duality mapping $J_\varphi$ with gauge function $\varphi$. Let $C$ be a nonempty, closed and convex subset of $X$ and let $T_1, T_2, \ldots, T_N$ be nonexpansive self-mappings of $C$ with $F := \bigcap_{i=1}^N \text{Fix}(T_i) \neq \emptyset$. Assume that

$$F = \text{Fix}(T_N \ldots T_1) = \text{Fix}(T_1T_N \ldots T_2) = \cdots = \text{Fix}(T_{n-1}T_{n-2} \cdots T_1T_N).$$

Let $\{\lambda_n\} \subset (0, 1)$ satisfy the following conditions:

[N1] $\lim_{n \to \infty} \lambda_n = 0$.

[N2] $\sum_{n=1}^\infty \lambda_n = \infty$.

[N3] $\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

Given points $x_0, y \in C$, the sequence $\{x_n\} \subset C$ is defined by

$$x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n \quad n \geq 0.$$ 

If there exists a sunny and nonexpansive retraction $P_F$ of $C$ onto $F$, then $x_n \to P_Fy$.

**Proof.** Following the idea of the proof in [10, Theorem 4.1], we divide the proof into several steps.

**Step 1.** $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$.

**Step 2.** $\{x_n\}$ is bounded.

**Step 3.** $\{T_{n+1}x_n\}$ is bounded.

**Step 4.** $x_{n+1} - T_{n+1}x_n \to 0$.

**Step 5.** $x_{n+N} - x_n \to 0$. 

Step 6. \[ x_n - T_{n+1}x_n \to 0. \]

Step 7. \[ \limsup_{n \to \infty} \langle y - P_F y, J_{\phi}(x_n - P_F y) \rangle \leq 0. \]

Step 8. \[ x_n \to P_F y. \]

At first it is easy to see that Steps 1-4 are the same as those in Theorem 3.1 and the proofs are thus omitted. Next we give the proofs of Steps 5-8, respectively.

Step 5: By Step 3, there exists a constant \( L > 0 \) such that for all \( n \geq 1 \),

\[ \| y - T_{n+1}x_n \| \leq L. \]

Since for all \( n \geq 1 \), \( T_{n+N} = T_n \), we have

\[
\| x_{n+N} - x_n \| = \| (\lambda_{n+N} - \lambda_n)(y - T_{n+N}x_{n+N-1}) \\
+ (1 - \lambda_{n+N})(T_nx_{n+N-1} - T_nx_{n-1}) \| \\
\leq L|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\| x_{n+N-1} - x_{n-1} \| \\
= (1 - \lambda_{n+N})\| x_{n+N-1} - x_{n-1} \| + \lambda_{n+N}L \left| 1 - \frac{\lambda_n}{\lambda_{n+N}} \right|. 
\]

By [N3] we have \( \lim_{n \to \infty} L\left| 1 - \frac{\lambda_n}{\lambda_{n+N}} \right| = 0 \) and so by Lemma 2.5,

\[ x_{n+N} - x_n \to 0. \]

Step 6: The proof of this step is taking from Step 4 in the proof of Theorem 3.2 [21]; see [21, p. 195]. Noting that each \( T_i \) is nonexpansive and using Step 4, we get the finite table

\[
x_{n+N} - T_{n+N}x_{n+N-1} \to 0, \\
T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \to 0, \\
\vdots \\
T_{n+N}...T_{n+2}x_{n+1} - T_{n+N}...T_{n+1}x_n \to 0. 
\]

Adding up this table yields

\[ x_n - T_{n+N}...T_{n+1}x_n \to 0. \]

Step 7: By Step 2, \( \{ \langle y - P_F y, J_{\phi}(x_n - P_F y) \rangle \} \) is bounded and hence

\[ \limsup_{n \to \infty} \langle y - P_F y, J_{\phi}(x_n - P_F y) \rangle \]
exists. Thus we can pick a subsequence \( \{n_i\} \) of \( \{n\} \) such that

\[
\limsup_{n \to \infty} (y - P_F y, J_\varphi(x_n - P_F y)) = \lim_{i \to \infty} (y - P_F y, J_\varphi(x_{n_i} - P_F y))
\]

and \( x_{n_i} \to p \) for some \( p \in C \).

The proof of \( p \in F \) given below is taking from Step 5 in the proof of Theorem 3.2 [21]; see [21, p. 195]. Since the pool of mappings \( \{T_i : 1 \leq i \leq N\} \) is finite, we may further assume (passing to a further subsequence if necessary) that for some integer \( k \in \{1, 2, ..., N\} \),

\[
n_i \mod N \equiv k, \quad \forall i \geq 1.
\]

Then it follows from Step 6 that

\[
x_{n_i} - T_{k+N}...T_{k+1}x_{n_i} \to 0.
\]

Hence by Lemma 2.1, we conclude that \( p \in Fix(T_{k+N}...T_{k+1}) \) which implies that \( p \in F \) from our assumption. Now by similar argument of Step 5 in the proof of Theorem 3.1, we can show that

\[
\limsup_{n \to \infty} (y - P_F y, J_\varphi(x_n - P_F y)) \leq 0.
\]

Finally Step 8 can be shown by the same argument of Step 6 in the proof of Theorem 3.1. The proof is now complete.

**ACKNOWLEDGMENT**

The authors thank the anonymous referees and Professor Adrian Petrușel for careful reading and constructive suggestions which led to helpful improvement of the paper.

**REFERENCES**


L. C. Ceng
Department of Mathematics,
Shanghai Normal University,
Shanghai 200234,
China.

P. Cubiotti
Department of Mathematics,
University of Messina,
Contrada Papardo,
Salita Sperone 31,
98166 Messina, Italy.

J. C. Yao*
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 804, Taiwan.
E-mail: yaojc@math.nsysu.edu.tw