q-GENERALIZATIONS OF THE PICARD AND GAUSS-WEIERSTRASS SINGULAR INTEGRALS

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Abstract. Introducing a higher order modulus of smoothness based on q-integers, in this paper first we obtain Jackson-type estimates in approximation by Jackson-type generalizations of the q-Picard and q-Gauss-Weierstrass singular integrals and give their global smoothness preservation property with respect to the uniform norm. Then, we study approximation and geometric properties of the complex variants for these q-singular integrals attached to analytic functions in compact disks. Finally, we prove approximation properties of these q-singular integrals attached to vector-valued functions.

1. Introduction

First we present some well known definitions and formulas for the q-calculus used throughout the paper.

For \( q > 0 \), the q-real \([\lambda]_q\), where \( \lambda \) is any real number, is defined

\[
[\lambda]_q := \begin{cases} 
\frac{1 - q^\lambda}{1 - q}, & q \neq 1 \\
1, & q = 1
\end{cases}
\]

and \([0]_q := 0\).

If \( \lambda \) is an integer, i.e. \( \lambda = n \) for some \( n \), we write \([n]_q\) and call it q–integer. Also, the q–factorial is defined as

\[
[n]_q ! := \begin{cases} 
[n]_q \cdot [n-1]_q \cdots [1]_q, & n = 1, 2, \ldots \\
1, & n = 0
\end{cases}
\]
The $q-$binomial coefficients are given by
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!},
\]
for integers $0 \leq k \leq n$, and as zero otherwise. Also, the $q-$binomial coefficients satisfy the following Pascal-type relation
\[
\binom{n}{k}_q = q^{n-k} \binom{n-1}{k-1}_q + \binom{n-1}{k}_q.
\]

The $q-$extension of exponential function $e^x$ is
\[
E_q(x) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} x^n = (-x; q)_\infty,
\]
where $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$ and $(-x; q)_\infty = \prod_{k=0}^{\infty} (1 + xq^k)$.

Furthermore, the $q$-binomial expansion is defined as
\[
\prod_{k=0}^{n-1} (1 + q^k x) = (-x; q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k.
\]

More details on these can be found in [16] and [15].

The following two integrals will play an important role throughout the paper. For $0 < q < 1$, the first integral, called the $q-$extension of Euler integral representation for the gamma function given in [13] and [2] that we use to define the $q-$Picard singular integral, is
\[
c_q(x) \Gamma_q(x) = \frac{1 - q}{\ln q^{-1} q^{x(1-x)}} \int_0^{\infty} \frac{t^{x-1}}{E_q((1 - q) t)} dt, \quad \Re x > 0
\]
where $\Gamma_q(x)$ is the $q-$gamma function defined by
\[
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad 0 < q < 1
\]
and $c_q(x)$ satisfies the following conditions: $c_q(x+1) = c_q(x)$, $c_q(n) = 1$, $n = 0, 1, 2, \ldots$ and $\lim_{q \to 1-} c_q(x) = 1$.

When $x = n + 1$ with $n$ a non-negative integer, we obtain
\[
\Gamma_q(n+1) = [n]_q!.
\]
The second integral that we use to define the $q-$Gauss-Weierstrass singular integral is given in [14], by

$$\int_{-\infty}^{\infty} \frac{t^{2k}}{E_q(t^2)} dt = \pi \left( \frac{q^{1/2}}{q} \right)^{1/2} \frac{q^{-k^2}}{q^{1/2}} \left( \frac{q^{1/2}}{q} \right)^k, \quad k = 0, 1, 2...$$

where we have $(a; q)_{\alpha} = (a; q)_{\infty} / (aq^\alpha; q)_{\infty}$, for any $\alpha \in \mathbb{R}$.

In [9], the first author generalizes the Picard and Gauss-Weierstrass singular integrals, to the so-called $q$-Picard and $q$-Gauss-Weierstrass singular integrals. In this paper, first we introduce $q$-Jackson type generalizations of these $q$-Picard and $q$-Gauss-Weierstrass singular integrals and obtain Jackson type error estimate in approximation and global smoothness preservation properties with respect to a $r$th $q$-uniform moduli of smoothness.

These results generalize and improve some results for classical Picard and Gauss-Weierstrass singular integrals and their Jackson type generalization in [3], [4], [5] and [17].

Then, we consider the complex versions of these $q$-singular integrals and study their approximation and geometric properties in the unit disk. The last section deals with approximation properties of these $q$-singular integrals attached to vector-valued functions.

2. $q$-JACKSON TYPE GENERALIZATION

First we give the $q$ analogous of the $r$th-modulus of smoothness of $f$ as it is defined in e.g. [17].

**Definition 1.** For $f \in C(\mathbb{R}), r \in \mathbb{N}$ and $q \in (0, 1)$ we introduce the following $r$th order $q$-moduli of smoothness of $f$ defined by

$$\omega_{r,q} (f; t) = \sup \{ |\Delta_{q,h}^r f (x)| : x, x + [r]_q h \in \mathbb{R}, 0 \leq h \leq t \},$$

where

$$\Delta_{q,h}^r f (x) = \sum_{k=0}^{r} (-1)^{r-k} q^{(r-k)(r-k-1)/2} \left[ \begin{array}{c} r \\ k \end{array} \right]_q f \left( x + [k]_q h \right).$$

The modulus $\omega_{1,q} (f; t)$ is denoted by $\omega (f; t)$ as in classical case.

Note that for $q = 1$ one reduces to the classical $r$th order moduli of smoothness defined as in e.g. [17] and [4, Chapter 2].

Reasoning as in the classical case (see e.g. [1]), we easily get
Lemma 1. For \( f \in C (\mathbb{R}) \) we have \( \omega_{r,q} (f; \gamma t) \leq (\gamma + 1)^r \omega_{r,q} (f; t) \).

Definition 2. Let \( f : \mathbb{R} \to \mathbb{R} \). For \( \lambda > 0, r \in \mathbb{N} \cup \{0\} \) and \( 0 < q < 1 \), the \( q \)-Jackson type generalization of \( q \)-Picard and \( q \)-Gauss-Weierstrass singular integrals of \( f \) are

\[
P_{r,\lambda} (f; q, x) \equiv P_{r,\lambda} (f; x) := -\frac{(1 - q)}{2 |\lambda| q \ln q} \sum_{k=1}^{r+1} (-1)^k \frac{q^{r-k+1}(r-k)/2}{q^{(r+1)/2}} \left( \begin{array}{c} r + 1 \\ k \end{array} \right) \int_{-\infty}^{\infty} f \left( x + \left[ \frac{k}{q} \right] t \right) dt.
\]

and

\[
W_{r,\lambda} (f; q, x) \equiv W_{r,\lambda} (f; x) := \frac{1}{\pi} \frac{1}{\sqrt{|\lambda| q (q^{1/2}; q)_{1/2}}} \sum_{k=1}^{r+1} (-1)^k \frac{q^{r-k+1}(r-k)/2}{q^{(r+1)/2}} \left( \begin{array}{c} r + 1 \\ k \end{array} \right) \int_{-\infty}^{\infty} f \left( x + \left[ \frac{k}{q} \right] t \right) \frac{q^{(r-k+1)}}{\lambda} dt.
\]

Note that for \( q = 1 \), the above definition one reduces to the classical Jackson-type generalization of Picard and Gauss-Weierstrass singular integrals of \( f \) defined in [17] and [4, Chapter 16], while for \( r = 0 \) we get the \( q \) singular integrals defined in [9].

Next we give approximation results with rates and global smoothness preservation properties.

Theorem 1. If \( f \in C (\mathbb{R}) \), \( r \in \mathbb{N} \cup \{0\} \) and \( 0 < q < 1 \), then we have

\[
|f(x) - P_{r,\lambda} (f; q, x)| \leq \omega_{r+1,q} (f; [\lambda]_q) \frac{1}{q^{(r+1)/2}} \sum_{k=0}^{r+1} \left( \begin{array}{c} r + 1 \\ k \end{array} \right) \frac{[k]_q!}{q^{2(k+1)}}
\]

and

\[
|f(x) - W_{2r-1,\lambda} (f; q, x)| \leq \omega_{2r,q} (f; [\lambda]_q) 2^{2r-1} \left( 1 + q^{-2} \left( \frac{q^{1/2}; q)_r}{\frac{1}{\lambda} q} \right) \right).
\]

Proof. Since \( \frac{(1 - q)}{2 |\lambda| q \ln q} \int_{-\infty}^{\infty} \frac{1}{E_q \left( \frac{1 - q}{|\lambda| q} \right)} dt = 1 \), we can write

\[
|f(x) - P_{r,\lambda} (f; q, x)| \leq \frac{(1 - q)}{2 |\lambda| q \ln q} \frac{1}{q^{(r+1)/2}} \int_{-\infty}^{\infty} \omega_{r+1,q} (f; [\lambda]_q) \frac{1}{E_q \left( \frac{1 - q}{|\lambda| q} \right)} dt.
\]
By the properties of the modulus of smoothness of a function given in Lemma 1, (1.4) and (1.5), we get

\[
|f(x) - P_{r \lambda}(f; q, x)| 
\leq \omega_{r+1,q}(f; [\lambda]_q) \frac{(1-q)}{[\lambda]_q \ln q^{-1}} \frac{1}{q^{(r+1)r/2}} \int_0^\infty \frac{\left(1+t/[\lambda]_q\right)^{r+1}}{E_q\left(1-q\frac{t}{[\lambda]_q}\right)} \, dt 
= \omega_{r+1,q}(f; [\lambda]_q) \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[k]_q!}{q^{k(k+1)/2}}.
\]

**Theorem 2.** Let \( f \in \mathcal{C}(\mathbb{R}) \), with \( \omega_{r,q}(f; \delta) < \infty \) for \( r \in \mathbb{N} \cup \{0\}, q \in (0, 1) \) and any \( \delta > 0 \). We have

\[
\omega_{r,q}(P_{r \lambda}f; \delta) \leq q^{-(r+1)r/2}((-1, q)_{r+1} - 1) \omega_{r,q}(f; \delta)
\]
and

\[
\omega_{r,q}(W_{r \lambda}f; \delta) \leq q^{-(r+1)r/2}((-1, q)_{r+1} - 1) \omega_{r,q}(f; \delta).
\]

**Proof.** We have for each \( 0 \leq h \leq \delta \)

\[
\Delta_{r,h}(P_{r \lambda}f)(x) = -\frac{(1-q)}{2 [\lambda]_q \ln q^{-1}} \sum_{k=1}^{r+1} (-1)^{r-k+1} q^{(r-k+1)(r-k)/2} \binom{r+1}{k} \int_{-\infty}^\infty \Delta_{r,k}^x f(x + [k]_q t) \frac{dt}{E_q(1-q \frac{t}{[\lambda]_q})}.
\]

By (1.3), we have desired result. The proof in the case of \( W_{r \lambda}(f; x) \) is similar.

### 3. COMPLEX \( q \)-PICARD AND \( q \)-GAUSS-WIEEERSTRASS INTEGRALS

In this section we extend the results in the case of classical complex Picard and Gauss-Weierstrass singular integrals proved in [6], [7], to their \( q \)-analogues.

Let us consider the open disk of radius \( R > 0 \), \( D_R = \{ z \in \mathbb{C} ; |z| < R \} \), \( A(D_R) = \{ f : D_R \to \mathbb{C} ; f \text{ is analytic on } D_R, \text{ continuous on } \overline{D_R} \} \), and \( A^*(D_R) = \{ f \in A(D_R) ; f(0) = 0, f'(0) = 1 \} \). Therefore, if \( f \in A^*(D_R) \) then we have \( f(z) = z + \sum_{k=2}^\infty a_k z^k \) for all \( z \in D_R \).
For $f \in A(D_R)$, $\lambda \in \mathbb{R}$, $\lambda > 0$, $0 < q < 1$, $r \in \mathbb{N} \cup \{0\}$ and $z \in \overline{D_R}$, let us define the $q$-complex singular integrals

$$P_{r\lambda}(f; z) \equiv P_{r\lambda}(f; z) := -\frac{(1-q)}{2|\lambda_q|^q} \ln q^{-1} \sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \left[ \frac{r+1}{k} \right] \int_{-\infty}^{\infty} f \left( \frac{ze^{i[k]qu}}{|\lambda_q|} \right) du,$$

and

$$W_{r\lambda}(f; z) \equiv W_{r\lambda}(f; z) := -\frac{1}{\pi \sqrt{|\lambda_q|^q(1/2; q)^{1/2}}} \sum_{k=1}^{r+1} (-1)^k \frac{q^{(r-k+1)(r-k)/2}}{q^{(r+1)r/2}} \left[ \frac{r+1}{k} \right] \int_{-\infty}^{\infty} f \left( \frac{ze^{i[k]qu}}{|\lambda_q|^q} \right) du,$$

called as the complex $q$-Jackson type generalization of the $q$-Picard and $q$-Gauss-Weierstrass singular integrals, respectively. For $r = 0$ we denote these singular integrals by $P_\lambda(f; z) \equiv P_\lambda(f; z)$ and $W_\lambda(f; z) \equiv W_\lambda(f; z)$, respectively.

First we present the approximation properties.

**Theorem 3.** Let $f \in A^*(D_R)$, i.e. $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in D_R$ with $a_0 = 0$, $a_1 = 1$ and $\lambda > 0$, $0 < q < 1$. We have:

(i) $P_\lambda(f; z) := P_\lambda(f; z)$ is continuous in $\overline{D_R}$, analytic in $D_R$ so that

$$P_\lambda(f; z) = \sum_{k=0}^{\infty} a_k c_k(\lambda, q) z^k, z \in D_R, P_\lambda(f; 0) = 0 \text{ and}$$

$$c_k(\lambda, q) = (1-q) \int_{0}^{\infty} \frac{\cos(ku)}{|\lambda_q|^q ln q^{-1}} E_{q} \left( \frac{(1-q)u}{|\lambda_q|^q} \right) du, k = 0,1,...$$

Also, there exists $\tilde{q} \in (0, 1)$ such that for all $q \in (\tilde{q}, 1)$ we have $c_1(\lambda, q) > 0$ and if we choose $q_\lambda$ such that $0 < q_\lambda < 1$ and $q_\lambda \to 1$ as $\lambda \to 0$, then we have $\lim_{\lambda \to 0} c_1(\lambda, q_\lambda) = 1$;

(ii) $|P_\lambda(f; z) - f(z)| \leq (R+1)(1+\frac{1}{q})\omega_1(f; |\lambda_q|)_{D_R}$ for all $z \in \overline{D_R}$, where

$$\omega_1(f; \delta)_{D_R} = \sup \{|f(z_1) - f(z_2)|; z_1, z_2 \in \overline{D_R}, |z_1 - z_2| \leq \delta\}.$$

**Proof.** (i) Let $z_0, z_n \in \overline{D_R}$ be with $\lim_{n \to \infty} z_n = z_0$. Since $|e^{iu}| = 1$, we get

$$|P_\lambda(f; z_n) - P_\lambda(f; z_0)| \leq$$
\[ \frac{(1 - q)}{2 [\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} |f(z_\infty e^{iu}) - f(z_0 e^{iu})| \cdot \frac{1}{E_q \left( \frac{(1-q)|u|}{[\lambda]_q} \right)} du \]

\[ \leq \frac{(1 - q)}{2 [\lambda]_q \ln q^{-1}} \int_{-\infty}^{\infty} \omega_1(f; |z_n - z_0|) du \cdot \frac{1}{E_q \left( \frac{(1-q)|u|}{[\lambda]_q} \right)} du = \omega_1(f; |z_n - z_0|) \]

Passing to limit with \( n \to \infty \), it follows that \( P_\lambda(f; z) \) is continuous at \( z_0 \in \overline{D}_R \). Since \( f \) is continuous on \( \overline{D}_R \). It remains to prove that \( P_\lambda(f; z) \) is analytic in \( D_R \).

For \( f \in A^\ast(D_R) \), we can write \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), \( z \in D_R \). For fixed \( z \in D_R \), we get \( f(z e^{iu}) = \sum_{k=0}^{\infty} a_k e^{iku} z^k \) and since \( |a_k e^{iku}| = |a_k| \), for all \( u \in \mathbb{R} \) and the series \( \sum_{k=0}^{\infty} a_k z^k \) is absolutely convergent, it follows that the series \( \sum_{k=0}^{\infty} a_k e^{iku} z^k \) is uniformly convergent with respect to \( u \in \mathbb{R} \). This immediately implies that the series can be integrated term by term, i.e.

\[ P_\lambda(f; z) = \frac{(1 - q)}{2 [\lambda]_q \ln q^{-1}} \sum_{k=0}^{\infty} a_k z^k \left( \int_{-\infty}^{\infty} e^{iku} \cdot \frac{1}{E_q \left( \frac{(1-q)|u|}{[\lambda]_q} \right)} du \right) \]

\[ = \sum_{k=0}^{\infty} a_k c_k(\lambda, q) z^k, \text{ where } c_k(\lambda, q) = \frac{(1 - q)}{[\lambda]_q \ln q^{-1}} \int_{0}^{\infty} \frac{e^{iku} |\cos(iku)|}{E_q \left( \frac{(1-q)|u|}{[\lambda]_q} \right)} du. \]

Since \( a_0 = 0 \), we get \( P_\lambda(f; 0) = 0 \).

Then we have

\[ c_1(\lambda, q) = \frac{(1 - q)}{[\lambda]_q \ln q^{-1}} \int_{0}^{\infty} \frac{\cos(u)}{E_q \left( \frac{(1-q)|u|}{[\lambda]_q} \right)} du = \frac{(1 - q)}{\ln q^{-1}} \int_{0}^{\infty} \frac{\cos(\lambda u)}{E_q ((1 - q) u)} du. \]

Now, if we choose \( q_\lambda \to 1 \) as \( \lambda \to 0 \), then we get \( [\lambda]_{q_\lambda} \to 0 \) (see [9]). Since \( \lim_{q \to 1} E_q((1-q) t) = e^t \) (see [16, p. 9, (1.3.16)]) and \( \lim_{q \to 1} [\lambda]_q = \lambda \), by Lebesgue’s Dominated Convergence theorem, we obtain

\[ \lim_{\lambda \to 0} c_1(\lambda, q_\lambda) = \int_{0}^{\infty} e^{-t} dt = 1 \]

\[ \lim_{q \to 1} c_1(\lambda, q) = \int_{0}^{\infty} \frac{\cos(\lambda u)}{e^u} du > ( \text{by e.g. [6, p.4]} ) > 0. \]

Thus, there exists \( \tilde{q} \in (0, 1) \) such that for all \( q \in (\tilde{q}, 1) \) we have \( c_1(\lambda, q) > 0 \).
(ii) By the Maximum Modulus Principle, it suffices to take $|z| = R$. Since $|e^{iu} - 1| \leq 2|\sin \frac{u}{2}| \leq |u|$ for all $u \in \mathbb{R}$, we easily get

$$|P_{\lambda}(f; z) - f(z)| \leq \frac{(1 - q)}{2[\lambda]_{q} \ln q^{-1}} \int_{-\infty}^{\infty} \lambda_{1}(f; |ze^{iu} - z|)_{D_{R}} \cdot \frac{1}{E_{q}\left(\frac{(1-q)|u|}{|\lambda|_{q}}\right)} \, du$$

$$\leq \frac{(1 - q)}{2[\lambda]_{q} \ln q^{-1}} \int_{-\infty}^{\infty} \omega_{1}(f; R|u|)_{D_{R}} \cdot \frac{1}{E_{q}\left(\frac{(1-q)|u|}{|\lambda|_{q}}\right)} \, du$$

$$\leq \omega_{1}(f; [\lambda]_{q})_{D_{R}}(R + 1) \frac{(1 - q)}{2[\lambda]_{q} \ln q^{-1}} \int_{-\infty}^{\infty} \left(1 + \frac{|u|}{|\lambda|_{q}}\right) \cdot \frac{1}{E_{q}\left(\frac{(1-q)|u|}{|\lambda|_{q}}\right)} \, du$$

$$\leq (\text{by [9]}) \leq (R + 1) \left(1 + \frac{1}{q}\right) \omega_{1}(f; [\lambda]_{q})_{D_{R}}. \quad \blacksquare$$

**Theorem 4.**

(i) If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $D_{R}$, then for all $\lambda > 0$, $0 < q < 1$, $W_{\lambda}(f; q, z) := W_{\lambda}(f; z)$ is analytic in $D_{R}$ and we have in $D_{R}$

$$W_{\lambda}(f; z) = \sum_{k=0}^{\infty} a_k d_k(\lambda, q) z^k,$$

where

$$d_k(\lambda, q) = \frac{2}{\pi \sqrt{|\lambda|_{q} (q^{1/2}; q)_{1/2}}} \int_{0}^{\infty} \cos(ku) E_{q}\left(\frac{u^2}{|\lambda|_{q}}\right) \, du.$$

Also, there exists $\hat{q} \in (0, 1)$ such that for all $q \in (\hat{q}, 1)$ we have $d_1(\lambda, q) > 0$ and if we choose $q_{\lambda}$ such that $0 < q_{\lambda} < 1$ and $q_{\lambda} \to 1$ as $\lambda \to 0$, then we have $\lim_{\lambda \to 0} d_1(\lambda, q_{\lambda}) = 1$.

In addition, if $f$ is continuous on $D_{R}$ then $W_{\lambda}(f; z)$ is continuous on $D_{R}$.

(ii) $|W_{\lambda}(f; z) - f(z)| \leq (R + 1) \left(1 + \sqrt{q^{-1/2}(1 - q^{1/2})}\right) \omega_{1}\left(f; \sqrt{|\lambda|_{q}}\right)_{D_{R}},$

for all $z \in D_{R}$.

**Proof.**

(i) Reasoning as for the $P_{\lambda}(f)$ operator, we easily deduce

$$W_{\lambda}(f; z) = \frac{1}{\pi \sqrt{|\lambda|_{q} (q^{1/2}; q)_{1/2}}} \int_{-\infty}^{+\infty} a_k z^k e^{iku} \cdot \frac{1}{E_{q}\left(\frac{u^2}{|\lambda|_{q}}\right)} \, du.$$
\[= \sum_{k=0}^{\infty} a_k d_k(\lambda, q) z^k, \text{ where } d_k(\lambda, q) = \frac{2}{\pi \sqrt{[\lambda]_q (q^{1/2}; q)_{1/2}}} \int_0^{+\infty} \cos(\lambda u) \frac{du}{E_q(\frac{u^2}{\lambda_q})} \]

Similar results with those for \(c_1(\lambda, q)\) (in Theorem 3), can be obtained for \(d_1(\lambda, q)\) too. Indeed, if we choose \(q_\lambda\) such that \(0 < q_\lambda < 1\) and \(q_\lambda \to 1\) as \(\lambda \to 0\), then from Lebesgue’s Dominated Convergence theorem, we get

\[
\lim_{\lambda \to 0} d_1(\lambda, q_\lambda) = \lim_{\lambda \to 0} \frac{2}{\pi \sqrt{[\lambda]_q (q^{1/2}; q)_{1/2}}} \int_0^{+\infty} \cos(\lambda u) \frac{du}{E_q(\frac{u^2}{\lambda_q})}
\]

\[
= \lim_{\lambda \to 0} \frac{2}{\pi (q^{1/2}; q)_{1/2}} \int_0^{+\infty} \frac{\cos(\sqrt{[\lambda]_q u})}{E_q(\frac{u^2}{\lambda_q})} du = (\text{ see e.g. [2, p.132]}) = 1
\]

Similarly we can see that \(\lim_{q \to 1} d_1(\lambda, q) > 0\), which implies that there exists \(\tilde{q} \in (0, 1)\) such that for all \(q \in (\tilde{q}, 1)\) we have \(d_1(\lambda, q) > 0\).

The proof of continuity of \(W_\lambda(f; z)\) is similar to that for \(P_\lambda(f; z)\).

(ii) Reasoning as in the case of \(P_\lambda(f; z)\), we can write

\[
|W_\lambda(f; z) - f(z)| \leq \frac{1}{\pi \sqrt{[\lambda]_q (q^{1/2}; q)_{1/2}}} \int_0^{+\infty} |f(ze^{-iu}) - f(z)| \frac{1}{E_q(\frac{u^2}{\lambda_q})} du
\]

\[
\leq \omega_1(f; [\lambda]_q)_{DR}(R + 1) \frac{1}{\pi \sqrt{[\lambda]_q (q^{1/2}; q)_{1/2}}} \int_0^{+\infty} \left(1 + \frac{|u|}{\sqrt{[\lambda]_q}}\right) \frac{1}{E_q(\frac{u^2}{\lambda_q})} du
\]

\[
\leq (\text{ see [9]}) \leq (R + 1) \left(1 + \sqrt{\frac{q^{-1/2}(1 - q^{1/2})}{2}}\right) \omega_1(f; [\lambda]_q)_{DR}.
\]

**Theorem 5.** For \(R > 0, z \in \overline{D}_R, \lambda \in (0, 1], 0 < q < 1\) and \(r \in \mathbb{N}\), we have

\[
|P_{r\lambda}(f; z) - f(z)| \leq \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{[k]_{q^r}!}{q^{\frac{(r+1)r}{2}}} \omega_{r+1,q}(f; [\lambda]_q)_{\partial D_R},
\]

\[
|W_{(2r-1)\lambda}(f; z) - f(z)| \leq 2^{2r-1} \left(1 + q^{-2} \left( q^{1/2}; q \right)_r \right) \omega_{2r,q}(f; [\lambda]_q)_{\partial D_R},
\]

where

\[
\omega_{r,q}(f; \delta)_{\partial D_R} = \sup\{|\Delta_{r,q}^* f(Re^{ix})|; |x| \leq \pi, |u| \leq \delta\}.
\]
Principle, it suffices to estimate \(|P_{r\lambda}(f; z) - f(z)|\), for this \(|z| = R, z = Re^{ix}\).

Reasoning now exactly as in the proof of Theorem 3, we get

\[
f(z) - P_{r\lambda}(f; z) = \frac{(1 - q)}{2 |\lambda|_q \ln q^{-1} q^{(r+1)r/2}} \int_{-\infty}^{\infty} \frac{\sum_{q,t} f \left( Re^{ix} \right)}{E_q \left( \frac{(1-q)|t|}{|\lambda|_q} \right)} dt,
\]

which implies

\[
|f(z) - P_{r\lambda}(f; z)| \leq \frac{(1 - q)}{2 |\lambda|_q \ln q^{-1} q^{(r+1)r/2}} \int_{-\infty}^{\infty} \frac{\omega_{r+1,q}(f; |t|)_{\partial D_R} dt}{E_q \left( \frac{(1-q)|t|}{|\lambda|_q} \right)}
\]

\[
\leq \omega_{r+1,q}(f; |\lambda|_q) \frac{1}{q^{(r+1)r/2}} \sum_{k=0}^{r+1} \binom{r+1}{k} \left( \frac{k q^{-1}}{q^{(r+1)r/2}} \right).
\]

The proof in the case of \(W_{2r-1}\lambda(f; z)\) is similar.

The geometric properties are consequences of Theorems 3 and 4 and are expressed by the following.

**Theorem 6.** Let us suppose that \(G \subset \mathbb{C}\) is open, such that \(\overline{D_1} \subset G\) and \(f: G \to \mathbb{C}\) is analytic in \(G\). Denote by \((B_{\lambda}(f)(z))_{\lambda > 0}\) any from \((P_{\lambda}(f; q, z))_{\lambda > 0}\), \((W_{\lambda}(f; q, z))_{\lambda > 0}\), where we choose \(q := q_\lambda\) such that \(0 < q_\lambda < 1\) and \(q_\lambda \to 1\) as \(\lambda \to 0\).

(i) If \(f\) is univalent in \(\overline{D_1}\), then there exists \(\lambda_0 > 0\) sufficiently small (depending on \(f\)), such that for all \(\lambda \in (0, \lambda_0)\), \(B_{\lambda}(f)(z)\) are univalent in \(\overline{D_1}\).

(ii) Let \(\gamma \in (-\pi/2, \pi/2)\). If \(f(0) = f'(0) - 1 = 0\) and \(f(z) \neq 0\), for all \(z \in \overline{D_1} \setminus \{0\}\) in the case of spirallikeness of order \(\gamma\) and \(f\) is starlike (convex, spirallike of order \(\gamma\), respectively) in \(\overline{D_1}\), that is for all \(z \in \overline{D_1}\)

\[
\text{Re} \left( z f'(z) \right) > 0, \text{Re} \left( \frac{z f''(z)}{f'(z)} \right) > 0, \text{Re} \left( e^{i\gamma} z f'(z) \right) > 0, \text{resp.}
\]

then there exists \(\lambda_0 > 0\) sufficiently small (depending on \(f\), and on \(f\) and \(\gamma\) in the case of spirallikeness), such that for all \(\lambda \in (0, \lambda_0)\), \(B_{\lambda}(f)(z)\) are starlike (convex, spirallike of order \(\gamma\), respectively) in \(\overline{D_1}\). If \(f(0) = f'(0) - 1 = 0\) and \(f(z) \neq 0\), for all \(z \in \overline{D_1} \setminus \{0\}\) in the case of spirallikeness of order \(\gamma\) and \(f\) is starlike (convex, spirallike of order \(\gamma\), respectively) only in \(D_1\) (that is the corresponding inequalities hold only in
then for any disk of radius \(0 < \rho < 1\) and center \(0\) denoted by \(D_\rho\), there exists \(\lambda_0 > 0\) sufficiently small (depending on \(f\) and \(D_\rho\), and in addition on \(\gamma\) for spirallikeness), such that for all \(\lambda \in (0, \lambda_0)\), \(B_\lambda(f)(z)\) are starlike (convex, spirallike of order \(\gamma\), respectively) in \(\overline{D}\rho\) (that is, the corresponding inequalities hold in \(\overline{D}\rho\)).

**Proof.** (i) Reasoning as in [9, Theorem 2.3], we get uniform convergence (as \(\lambda \to 0\)) in Theorems 3 and 4, which together with a well-known results concerning sequences of analytic functions converging locally uniformly to an univalent function (see e.g. [20], p. 130, Theorem 4.1.17) implies the univalence of \(B_\lambda(f)(z)\) for sufficiently small \(\lambda\).

For the proof of the conclusions in (ii), let us make some general useful considerations. By Theorems 3 and 4 (reasoning again as in [9, Theorem 2.3]), it follows that for \(\lambda \to 0\), we have \(B_\lambda(f)(z) \to f(z)\), uniformly in any compact disk included in \(G\). By the well-known Weierstrass’ result (see e.g. [20], p. 18, Theorem 1.1.6), this implies that \(B_\lambda'(f)(z) \rightarrow f'(z)\) and \(B''_\lambda(f)(z) \rightarrow f''(z)\), uniformly in any compact disk in \(G\) and therefore in \(\overline{D}\lambda\) too, when \(\lambda \to 0\). In all what follows, denote \(P_\lambda(f)(z) = \frac{B_\lambda(f)(z)}{b_1(\lambda, q_\lambda)}\), where \(b_1(\lambda, q_\lambda) > 0\) (for \(\lambda\) sufficiently small) is the coefficient of \(z\) in the Taylor series representing the analytic function \(B_\lambda(f)(z)\).

If \(f(0) = f'(0) − 1 = 0\), then we get \(P_\lambda(f)(0) = \frac{f(0)}{b_1(\lambda, q_\lambda)} = 0\) and \(P_\lambda'(f)(0) = \frac{b_1'/(f(0))}{b_1(\lambda, q_\lambda)} = 1\). Also, if \(f(0) = 0\) and \(f'(0) = 1\), then \(b_1(\lambda, q_\lambda)\) converges to \(f'(0) = 1\) as \(\lambda \to 0\), which obviously implies that for \(\lambda \to 0\), we have \(P_\lambda(f)(z) \to f(z)\), \(P_\lambda'(f)(z) \to f'(z)\) and \(P_\lambda''(f)(z) \to f''(z)\), uniformly in \(\overline{D}\lambda\).

(ii) Suppose first that \(f\) is starlike in \(\overline{D}\lambda\). By hypothesis we get \(|f(z)| > 0\) for all \(z \in \overline{D}\lambda\) with \(z \neq 0\), which from the univalence of \(f\) in \(D\lambda\), implies that we can write \(f(z) = zg(z)\), with \(g(z) \neq 0\), for all \(z \in \overline{D}\lambda\), where \(g\) is analytic in \(D\lambda\) and continuous in \(\overline{D}\lambda\).

Write \(P_\lambda(f)(z)\) in the form \(P_\lambda(f)(z) = zQ_\lambda(f)(z)\). For \(|z| = 1\) we have

\[
|f(z) − P_\lambda(f)(z)| = |z| \cdot |g(z) − Q_\lambda(f)(z)| = |g(z) − Q_\lambda(f)(z)|,
\]

which by the uniform convergence in \(\overline{D}\lambda\) of \(P_\lambda(f)\) to \(f\) and by the maximum modulus principle, implies the uniform convergence in \(\overline{D}\lambda\) of \(Q_\lambda(f)(z)\) to \(g(z)\), as \(\lambda \to 0\).

Since \(g\) is continuous in \(\overline{D}\lambda\) and \(|g(z)| > 0\) for all \(z \in \overline{D}\lambda\), there exist an index \(\lambda_0 > 0\) and \(\alpha > 0\) depending on \(g\), such that \(|Q_\lambda(f)(z)| > \alpha > 0\), for all \(z \in \overline{D}\lambda\) and all \(\lambda \in (0, \lambda_0)\). Also, for all \(|z| = 1\), we have

\[
|f'(z) − P_\lambda'(f)(z)| = |z| |g'(z) − Q_\lambda'(f)(z)| + |g(z) − Q_\lambda(f)(z)|
\]

\[
\geq |z| \cdot |g'(z) − Q_\lambda'(f)(z)| − |g(z) − Q_\lambda(f)(z)| \quad |\quad = |g'(z) − Q_\lambda'(f)(z)| − |g(z) − Q_\lambda(f)(z)| \quad |,
\]
which from the maximum modulus principle, the uniform convergence of $P'_\lambda(f)$ to $f'$ and of $Q_\lambda(f)$ to $g$, evidently implies the uniform convergence of $Q'_\lambda(f)$ to $g'$, as $\lambda \to 0$. Then, for $|z| = 1$, we get

$$\frac{zP'_\lambda(f)(z)}{P_\lambda(f)} = \frac{z[zQ'_\lambda(f)(z) + Q_\lambda(f)(z)]}{zQ_\lambda(f)(z)} = \frac{zQ'_\lambda(f)(z) + Q_\lambda(f)(z)}{Q_\lambda(f)(z)} \to \frac{zg'(z) + g(z)}{g(z)} = \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)}.$$ 

which again from the maximum modulus principle, implies

$$\frac{zP'_\lambda(f)(z)}{P_\lambda(f)} \to \frac{zf'(z)}{f(z)},$$

uniformly in $\overline{D_1}$.

Since $\text{Re} \left( \frac{zf'(z)}{f(z)} \right)$ is continuous in $\overline{D_1}$, there exists $\alpha \in (0, 1)$, such that

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq \alpha, \text{ for all } z \in \overline{D_1}.$$ 

Therefore

$$\text{Re} \left[ \frac{zP'_\lambda(f)(z)}{P_\lambda(f)(z)} \right] \to \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] \geq \alpha > 0$$

uniformly on $\overline{D_1}$, i.e. for any $0 < \beta < \alpha$, there is $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ we have

$$\text{Re} \left[ \frac{zP'_\lambda(f)(z)}{P_\lambda(f)(z)} \right] > \beta > 0, \text{ for all } z \in \overline{D_1}.$$ 

Since $P_\lambda(f)(z)$ differs from $B_\lambda(f)(z)$ only by a constant, this proves the starlikeness in $\overline{D_1}$.

If $f$ is supposed to be starlike only in $D_1$, the proof is identical, with the only difference that instead of $\overline{D_1}$, we reason for $\overline{D_\rho}$.

The proofs in the cases when $f$ is convex or spirallike of order $\gamma$ are similar and follows from the following uniform convergences (on $\overline{D_\rho}$ or on $\overline{D_\rho}$)

$$\text{Re} \left[ \frac{zf''(z)}{f'(z)} \right] + 1 \to \text{Re} \left[ \frac{zf'(z)}{f'(z)} \right] + 1.$$ 

and

$$\text{Re} \left[ e^{i\gamma} \frac{zP'_\lambda(f)(z)}{P_\lambda(f)(z)} \right] \to \text{Re} \left[ e^{i\gamma} \frac{zf'(z)}{f(z)} \right].$$

The proof is complete.
**Remark 1.** By using Theorem 5 and reasoning as above, it is not difficult to prove that the geometric properties in Theorem 6 remain valid for $P_{r\lambda}(f; z)$ and $W_{r\lambda}(f; z)$ too.

4. *q*-SINGULAR INTEGRALS ATTACHED TO VECTOR VALUED FUNCTIONS

In this section we extend some of the above results to vector-valued functions. Note that the case of classical singular integrals attached to vector valued functions was considered in [7].

If $(X, \| \cdot \|)$ is a complex Banach space and $R > 0$, let us denote by $A(D_R; X)$ the space of all functions $f : D_R \to X$, which are continuous in $D_R$ and holomorphic in $D_R$. Recall that according to e.g. [19], p. 97), any $f \in A(D_R; X)$ has the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad z \in D_R,$$

where the series converges uniformly on any compact subset of $D_R$.

We will use the following well-known result in Functional Analysis.

**Theorem 7.** Let $(X, \| \cdot \|)$ be a normed space over $\mathbb{R}$ of $\mathbb{C}$ and denote by $X^*$ the conjugate of $X$. Then $\|x\| = \sup \{|x^*(x)|; x^* \in X^*, \|x^*\| \leq 1\}$, for all $x \in X$, where $\| \cdot \|$ represents the usual norm in the dual space $X^*$.

Now we are in position to prove our result. We present

**Theorem 8.** Let $f \in A(D_R; X), (X, \| \cdot \|)$ a complex normed space. If for $\lambda > 0$, $0 < q < 1$, we consider the operators

$$P\lambda (f; q, z) \equiv P_{\lambda}(f; z) := \frac{(1-q)}{2 \lambda q \ln q} \int_{-\infty}^{\infty} \frac{f(ze^{it})}{E_q \left( \frac{(1-q)|t|}{\lambda q} \right)} dt,$$

$$W\lambda (f; q, z) \equiv W_{\lambda}(f; z) := \frac{1}{\pi \sqrt{\lambda q}} \int_{-\infty}^{\infty} \frac{f(ze^{it})}{E_q \left( \frac{q^{1/2}}{\lambda q} \right)} dt,$$

then we have

$$\|P_{\lambda}(f; z) - f(z)\| \leq (R+1)(1+ \frac{1}{q}) \omega_1(f; [\lambda q]_{D_R}),$$

$$\|W_{\lambda}(f; z) - f(z)\| \leq (R+1) \left(1 + \sqrt{q^{-1/2}(1-q^{-1/2})} \right) \omega_1 \left(f; \sqrt{[\lambda q]}_{D_R} \right),$$
for all \( z \in \overline{D_R} \), where \( \omega_1(f; \delta)_{DR} = \sup\{||f(z_1) - f(z_2)||; z_1, z_2 \in \overline{D_R}, |z_1 - z_2| \leq \delta \} \).

\textbf{Proof.} Let \( x^* \in B_1 \) and define \( g(z) = x^*[f(z)], g: \overline{D_R} \to \mathbb{C} \). By Theorem 3 we have \(|P_\lambda(g; z) - g(z)| \leq 2(1 + \frac{1}{q})\omega_1(g; |\lambda|_q)_{DR} \), for all \( z \in \overline{D_R} \), where

\[
\omega_1(g; \delta)_{DR} = \sup\{||x^*[f(z_1) - f(z_2)]||; z_1, z_2 \in \overline{D_R}, |z_1 - z_2| \leq \delta \} \leq \sup\{||f(z_1) - f(z_2)||; z_1, z_2 \in \overline{D_R}, |z_1 - z_2| \leq \delta \} = \omega_1(f; \delta)_{DR}.
\]

Therefore, we obtain \(|x^*[P_\lambda(f; z) - f(z)]| \leq 2(1 + \frac{1}{q})\omega_1(f; |\lambda|_q)_{DR} \), for all \( x^* \in B_1 \), and passing here to supremum, according to Theorem 7 it follows the required estimate. The proof in the case of \( W_\lambda(f; z) \) is similar. \( \blacksquare \)

\textbf{Remark 2.} By using the method in the proof of Theorem 8, analogous results can easily be proved for \( P_r(\lambda; f; z) \) and \( W_r(\lambda; f; z) \).

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