BOUNDEDNESS OF SUBLINEAR OPERATORS IN HERZ-TYPE HARDY SPACES

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Abstract. Let $p \in (0, 1], q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$ and $w_1, w_2 \in A_1$. The author proves that the norms in weighted Herz-type Hardy spaces $HK^q_{\alpha, p}(w_1, w_2)$ and $HK^q_{\alpha, p}(w_1, w_2)$ can be achieved by finite central atomic decompositions in some dense subspaces of them. As an application, the author proves that if $T$ is a sublinear operator and maps all central $(\alpha, q, s; w_1, w_2)$-atoms (resp. central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type) into uniformly bounded elements of certain quasi-Banach space $B$ for certain nonnegative integer $s$ no less than the integer part of $\alpha - n(1 - 1/q)$, then $T$ uniquely extends to a bounded operator from $HK^q_{\alpha, p}(w_1, w_2)$ (resp. $HK^q_{\alpha, p}(w_1, w_2)$) to $B$.

1. INTRODUCTION

The theory of Hardy spaces associated to Herz spaces obtained a great development in the past few years and played important roles in Harmonic analysis; see [4, 1, 14, 15, 16, 17, 7, 8, 11].

To establish the boundedness of operators in Hardy type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization (see [5, 13, 14]) of these spaces, which means that a function or distribution in Hardy type spaces can be represented as a linear combination of atoms. Then, the boundedness of linear operators in Hardy type spaces can be deduced from their behavior on atoms in principle.

However, Meyer [20, p. 513] (see also [6, 3]) gave an example of $f \in H^1(\mathbb{R}^n)$ whose norm cannot be achieved by its finite atomic decompositions via $(1, \infty, 0)$-atoms. Based on this fact, Bownik [3, Theorem 2] constructed a surprising example of a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all...
some Banach space $B$ by $Y$. Meyer in [19, p. 19]. Moreover, motivated by this, Yabuta to Hardy spaces bounded elements of $B$ for some $s \geq \lfloor n(1/p - 1) \rfloor$. Here and in what follows $\lfloor t \rfloor$ means the integer part of real $t$. We should point out that this phenomenon has essentially already been observed by Y. Meyer in [19, p. 19]. Moreover, motivated by this, Yabuta [23] gave some very general sufficient conditions for the boundedness of a linear operator $T$ from $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ to some quasi-Banach space $B$ only proving that $T$ maps all $(p, \infty, s)$-atoms into uniformly bounded elements of $B$ for some $s \geq \lfloor n(1/p - 1) \rfloor$. In [12], Yabuta’s results were generalized to the setting of spaces of homogeneous type, and moreover, a sufficient condition for the boundedness of $T$ from $H^p$ with $p \in (0, 1)$ to $L^q$ with $q \in [p, 1]$ is also provided. However, these conditions are not necessary.

Recently, in [24], a boundedness criterion was established via Lusin function characterizations of Hardy spaces on $\mathbb{R}^n$ as follows: a sublinear operator $T$ extends to a bounded sublinear operator from Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ to some quasi-Banach space $B$ if and only if $T$ maps all $(p, 2, s)$-atoms into uniformly bounded elements of $B$ for some $s \geq \lfloor n(1/p - 1) \rfloor$, which was also generalized to Hardy spaces $H^p$ with $p \in (0, 1)$ from below on spaces of homogeneous type having the reverse doubling property in [25]. This result shows the structural difference between atomic characterization of $H^p(\mathbb{R}^n)$ via $(p, 2, s)$-atoms and $(p, \infty, s)$-atoms. On the other hand, Meda, Sjögren and Vallarino [18] independently obtained some similar results by grand maximal function characterizations of Hardy spaces on $\mathbb{R}^n$. In fact, let $p \in (0, 1]$, $p < q \in [1, \infty]$ and integer $s \geq \lfloor n(1/p - 1) \rfloor$, and let $H^p_{\text{fin}}(\mathbb{R}^n)$ be the set of all finite linear combinations of $(p, q, s)$-atoms. Denote by $C(\mathbb{R}^n)$ the set of all continuous functions. For any $f \in H^p_{\text{fin}}(\mathbb{R}^n)$ when $q < \infty$ or $f \in H^p_{\text{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q = \infty$, Meda, Sjögren and Vallarino [18] proved that $\|f\|_{H^p(\mathbb{R}^n)}$ can be achieved by a finite atomic decomposition via $(p, q, s)$-atom when $q < \infty$ or continuous $(p, q, s)$-atom when $q = \infty$; from this, they further deduced that if $T$ is a linear operator and maps all $(1, q, 0)$-atoms with $q \in (1, \infty)$ or all continuous $(1, q, 0)$-atoms with $q = \infty$ into uniformly bounded elements of some Banach space $B$, then $T$ uniquely extends to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $B$ which coincides with $T$ on these $(1, q, 0)$-atoms. Grafos, Liu and Yang [9] generalize these results to Hardy spaces $H^p$ with $p \in (0, 1]$ from below on spaces of homogeneous type having the reverse doubling property.

In this paper, let $p \in (0, 1]$, $q \in (1, \infty)$, $\alpha \in n(1 - 1/q, \infty)$ and $w_1, w_2 \in A_1$. We prove that the norms in weighted Herz-type Hardy spaces $\dot{H}K_q^{\alpha,p}(w_1, w_2)$ and $HK_q^{\alpha,p}(w_1, w_2)$ can be achieved by finite central atomic decompositions in some dense subspaces of them. As an application, we prove that if $T$ is a sublinear operator and maps all central $(\alpha, q, s; w_1, w_2)$-atoms (resp. central $(\alpha, q, s; w_1, w_2)$-
atoms of restrict type) into uniformly bounded elements of certain quasi-Banach space $\mathcal{B}$ for certain nonnegative integer $s \geq \lfloor \alpha - n(1 - 1/q) \rfloor$, then $T$ uniquely extends to a bounded sublinear operator from $\dot{H}^\alpha_{q,p}(w_1, w_2)$ (resp. $H^\alpha_{q,p}(w_1, w_2)$) to $\mathcal{B}$.

This paper is organized as follows. In Section 2, we recall notations of weighted Herz-type Hardy spaces, and their central atomic decomposition characterizations in [14] via central $(\alpha, q, s; w_1, w_2)$-atoms or central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type. Moreover, we introduce central $(\alpha, q, s; w_1, w_2)$-0-atoms. In Section 3, we prove that the norms of weighted Herz-type Hardy spaces in some dense subspaces can be achieved by finite atomic decompositions via central $(\alpha, q, s; w_1, w_2)$-0-atoms and central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type; see Theorem 1 below. Then as an application, we establish some criteria to obtain the boundedness of sublinear operators in weighted Herz-type Hardy spaces (see Theorem 2 below).

Finally, we make some conventions. Throughout this paper, let $\mathbb{N}$ be the set all positive integer and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We always use $C$ to denote a positive constant that is independent of main parameters involved but whose value may differ from line to line. We use $f \lesssim g$ to denote $f \leq Cg$.

2. PRELIMINARIES

We begin with recalling definitions of weight Herz spaces; see [4, 14]. In what follows, we always let $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$ for any $r > 0$ and $x \in \mathbb{R}^n$, $B_k \equiv B(0, 2^k)$, $C_k \equiv B_k \setminus B_{k-1}$, $R_k \equiv (C_k \cup C_{k+1})$ and $\chi_k \equiv \chi_{C_k}$ for all $k \in \mathbb{Z}$.

**Definition 1.** Let $p \in (0, \infty)$, $q \in (1, \infty)$ and $\alpha \in \mathbb{R}$. Let $w_1$ and $w_2$ be nonnegative weight functions.

(i) The homogeneous weighted Herz space $\dot{K}^\alpha_{q,p}(w_1, w_2)$ is defined to be the set of all $f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; w_2)$ such that

$$\|f\|_{\dot{K}^\alpha_{q,p}(w_1, w_2)} \equiv \left\{ \sum_{k \in \mathbb{Z}} [w_1(B_k)]^{p\alpha/n} \|f\chi_k\|_{L^q(\mathbb{R}^n; w_2)}^p \right\}^{1/p} < \infty.$$ 

(ii) The non-homogeneous weighted Herz space $K^\alpha_{q,p}(w_1, w_2)$ is defined to be the set of all $f \in L^q_{\text{loc}}(\mathbb{R}^n; w_2)$ such that

$$\|f\|_{K^\alpha_{q,p}(w_1, w_2)} \equiv \left\{ \|f\chi_{B_0}\|_{L^q(\mathbb{R}^n; w_2)}^p + \sum_{k=1}^{\infty} [w_1(B_k)]^{p\alpha/n} \|f\chi_k\|_{L^q(\mathbb{R}^n; w_2)}^p \right\}^{1/p} < \infty.$$
If \( w_1 \equiv w_2 \equiv 1 \), then \( \dot{K}^{\alpha, p}_q(w_1, w_2) \) and \( K^{\alpha, p}_q(w_1, w_2) \) are the standard Herz spaces in [4] and also [14].

To define the corresponding Hardy spaces, we first recall some notation. Let \( \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\} \) and \( \mathcal{S}(\mathbb{R}^n) \) be the space of Schwartz functions endowed with the semi-norms \( \{\| \cdot \|_{m, \beta}\}_{m, \beta} \), where \( \| \phi \|_{m, \beta} \equiv \sup_{x \in \mathbb{R}^n} (1 + |x|^m) |D^\beta \phi(x)| \), \( \beta \equiv (\beta_1, \ldots, \beta_n) \) and \( D^\beta \phi \equiv (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n} \). Denote by \( \mathcal{S}'(\mathbb{R}^n) \) the dual space of \( \mathcal{S}(\mathbb{R}^n) \).

Let \( N \in \mathbb{N} \) and \( \mathcal{S}_N(\mathbb{R}^n) \equiv \{ \phi \in \mathcal{S}(\mathbb{R}^n) : \| \phi \|_{m, \beta} \leq 1, \ m \leq n + N, \ |\beta| \leq N \} \). For \( f \in \mathcal{S}'(\mathbb{R}^n) \), in [10], the grand maximum function of \( f \) is defined by \( G_N(f)(x) \equiv \sup_{\phi \in \mathcal{S}_N} M \phi(f)(x) \), where \( M \phi(f)(x) \equiv \sup_{|y-x|<t} |\phi_t * f(y)| \) and \( \phi_t(x) \equiv t^{-n} \phi(t^{-1} x) \) for all \( t > 0 \) and \( x \in \mathbb{R}^n \).

Recall in [6] that a function \( w \) is said to be in the Muckenhoupt class \( A_1 \) if there exists a constant \( C > 0 \) such that \( Mw(x) \leq Cw(x) \) for almost everywhere \( x \in \mathbb{R}^n \), where \( M \) is the Hardy-Littlewood maximal operator.

The Hardy spaces associated to the weighted Herz spaces in [14] are defined as below.

**Definition 2.** Let \( p \in (0, \infty), q \in (1, \infty), \alpha \in (0, \infty) \) and \( N \equiv \max\{\lfloor \alpha - n(1 - 1/q) \rfloor + 1, 1\} \). Let \( w_1, w_2 \in A_1 \).

(i) The homogeneous Hardy space \( H\dot{K}^{\alpha, p}_q(w_1, w_2) \) associated to \( \dot{K}^{\alpha, p}_q(w_1, w_2) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{H\dot{K}^{\alpha, p}_q(w_1, w_2)} \equiv \|G_N(f)\|_{\dot{K}^{\alpha, p}_q(w_1, w_2)} < \infty.
\]

(ii) The non-homogeneous Hardy space \( HK^{\alpha, p}_q(w_1, w_2) \) associated to \( K^{\alpha, p}_q(w_1, w_2) \) is defined to be the set of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{HK^{\alpha, p}_q(w_1, w_2)} \equiv \|G_N(f)\|_{K^{\alpha, p}_q(w_1, w_2)} < \infty.
\]

Let \( p \in (0, \infty), q \in (1, \infty) \) and \( w_1, w_2 \in A_1 \). Notice that if \( \alpha \in (0, n(1-1/q)) \), then \( (\dot{H}\dot{K}^{\alpha, p}_q(w_1, w_2) \cap L^q(\mathbb{R}^n \setminus \{0\}; w_2)) = \dot{K}^{\alpha, p}_q(w_1, w_2) \) and \( (HK^{\alpha, p}_q(w_1, w_2) \cap L^q(\mathbb{R}^n; w_2)) = \dot{K}^{\alpha, p}_q(w_1, w_2) \); and if \( \alpha \in [n(1-1/q), \infty) \), then \( \dot{H}K^{\alpha, p}_q(w_1, w_2) \subseteq \dot{K}^{\alpha, p}_q(w_1, w_2) \) and \( HK^{\alpha, p}_q(w_1, w_2) \subseteq \dot{K}^{\alpha, p}_q(w_1, w_2) \), see [14, 15]. Thus, in what follows, we always assume that \( p \in (0, \infty) \), \( q \in (1, \infty) \) and \( \alpha \in [n(1-1/q), \infty) \).

Now we state the definition of central atoms. Lu and Yang [14] introduced central \((\alpha, q, s; w_1, w_2)\)-atoms and central \((\alpha, q, s; w_1, w_2)\)-atoms of restrict type, and use them to characterize the spaces \( \dot{H}\dot{K}^{\alpha, p}_q(w_1, w_2) \) and \( HK^{\alpha, p}_q(w_1, w_2) \).

**Definition 3.** Let \( q \in (1, \infty), \alpha \in [n(1-1/q), \infty), s \geq |\alpha - n(1-1/q)| \) and \( w_1, w_2 \in A_1 \).
(i) A function $a$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s; w_1, w_2)$-atom if it satisfies that

(A1) $\text{supp} \ a \subset B(0, r)$ for some $r > 0$;

(A2) $\|a\|_{L^s(\mathbb{R}^n; w_2)} \leq |w_1(B(0, r))|^{-\alpha/n}$;

(A3) $\int_{\mathbb{R}^n} a(x)x^\beta \, dx = 0$ for all $|\beta| \leq s$.

(ii) A function $a$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s; w_1, w_2)_0$-atom if it satisfies (A1) through (A3), and $a(x) = 0$ on some neighborhood of $0$.

(iii) A function $a$ on $\mathbb{R}^n$ is called a central $(\alpha, q, s; w_1, w_2)$-atom of restrict type if it satisfies (A1) with $r \geq 1$, (A2) and (A3).

Theorem A. Let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$ and nonnegative integer $s \geq |\alpha - n(1 - 1/q)|$.

(i) Then $f \in H K_q^{\alpha,p}(w_1, w_2)$ if and only if $f = \sum_{k \in \mathbb{Z}^n} \lambda_k a_k$ in $\mathcal{S}'(\mathbb{R}^n)$, where $a_k$ is a central $(\alpha, q, s; w_1, w_2)$-atom supported in $B_k$, and $\sum_{k \in \mathbb{Z}^n} |\lambda_k|^p < \infty$. Moreover, $\|f\|_{H K_q^{\alpha,p}(w_1, w_2)} \sim \inf \{\sum_{k \in \mathbb{Z}^n} |\lambda_k|^p\}$, where the infimum is taken over all the above decompositions of $f$.

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3. Main Results and Their Proofs

In this section, we first prove that the norms in $H K_q^{\alpha,p}(w_1, w_2)$ and $H K_q^{\alpha,p}(w_1, w_2)$ can be achieved by finite central atomic decomposition in some dense subspaces of them.

To this end, let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$, $s \geq |\alpha - n(1 - 1/q)|$, and $w_1, w_2 \in A_1$. Denote by $F_p^{\alpha,q,s}(w_1, w_2)$ the collection of all finite linear combinations of central $(\alpha, q, s; w_1, w_2)$-atoms, and for $f \in F_p^{\alpha,q,s}(w_1, w_2)$, define

$$\|f\|_{F_p^{\alpha,q,s}(w_1, w_2)} \equiv \inf \left\{ \left( \sum_{j=1}^m |\lambda_j|^p \right)^{1/p} : m \in \mathbb{N}, f = \sum_{j=1}^m \lambda_j a_j, \{a_j\}_{j=1}^m \text{ are central } (\alpha, q, s; w_1, w_2)_0\text{-atoms} \right\}.$$  

Let $C F_p^{\alpha,q,s}(w_1, w_2)$ be the collection of all finite linear combinations of $C^\infty(\mathbb{R}^n)$ central $(\alpha, q, s; w_1, w_2)_0$-atoms, and for $f \in C F_p^{\alpha,q,s}(w_1, w_2)$, define $\|f\|_{C F_p^{\alpha,q,s}(w_1, w_2)}$.
as in (3.1) just replacing central $(\alpha, q, s; w_1, w_2)_0$-atoms by $C^\infty(\mathbb{R}^n)$ central $(\alpha, q, s; w_1, w_2)_0$-atoms.

We also denote by $F_p^\alpha,q,s(w_1, w_2)$ (resp. $CF_p^\alpha,q,s(w_1, w_2)$) the collection of all finite linear combinations of central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type (resp. $C^\infty(\mathbb{R}^n)$ central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type) and for $f \in F_p^\alpha,q,s(w_1, w_2)$ (resp. $f \in CF_p^\alpha,q,s(w_1, w_2)$), define $\|f\|_{F_p^\alpha,q,s(w_1, w_2)}$ (resp. $\|f\|_{CF_p^\alpha,q,s(w_1, w_2)}$) as (3.1) just replacing central $(\alpha, q, s; w_1, w_2)_0$-atom by central $(\alpha, q, s; w_1, w_2)$-atom of restrict type (resp. $C^\infty(\mathbb{R}^n)$ central $(\alpha, q, s; w_1, w_2)$-atom of restrict type).

For $s \in \mathbb{Z}_+$, let $D_s(\mathbb{R}^n)$ be the collection of all functions $f \in C^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} f(x) x^\beta dx = 0$ for all $|\beta| \leq s$, and let $D_s(\mathbb{R}^n)$ be the set of all functions $f \in D_s(\mathbb{R}^n)$ with $0 \notin \text{supp } f$.

One of our main results is as follows.

**Theorem 1.** Let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1-1/q), \infty)$ and nonnegative integer $s \geq |\alpha - n(1-1/q)|$.

(i) Then $\| \cdot \|_{HK^\alpha,q,w_1,w_2}$ and $\| \cdot \|_{F_p^\alpha,q,s(w_1, w_2)}$ (resp. $\| \cdot \|_{CF_p^\alpha,q,s(w_1, w_2)}$) are equivalent on $F_p^\alpha,q,s(w_1, w_2)$ (resp. $CF_p^\alpha,q,s(w_1, w_2)$).

(ii) Then $\| \cdot \|_{HK^\alpha,q,w_1,w_2}$ and $\| \cdot \|_{F_p^\alpha,q,s(w_1, w_2)}$ (resp. $\| \cdot \|_{CF_p^\alpha,q,s(w_1, w_2)}$) are equivalent on $F_p^\alpha,q,s(w_1, w_2)$ (resp. $CF_p^\alpha,q,s(w_1, w_2)$).

**Proof.** Since the proof of (ii) is similar to that of (i), we only prove (i) here. Moreover, we only prove the equivalence between $\| \cdot \|_{HK^\alpha,q,w_1,w_2}$ and $\| \cdot \|_{F_p^\alpha,q,s(w_1, w_2)}$ on $CF_p^\alpha,q,s(w_1, w_2)$ since the equivalence between $\| \cdot \|_{HK^\alpha,q,w_1,w_2}$ and $\| \cdot \|_{F_p^\alpha,q,s(w_1, w_2)}$ on $CF_p^\alpha,q,s(w_1, w_2)$ can be obtained by a slight modification.

It is easy to see that for any $f \in CF_p^\alpha,q,s(w_1, w_2)$, $\|f\|_{CF_p^\alpha,q,s(w_1, w_2)} \leq \|f\|_{HK^\alpha,q,w_1,w_2}$. To prove the converse, we use some ideas in [14].

Let $\psi \in S(\mathbb{R}^n)$ such that $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}^n$, $\psi(x) = 1$ if $|x| \leq 1/2 + 1/10$ and $\psi(x) = 0$ if $|x| \geq 1 - 1/10$. Let $\varphi(x) = \psi(x/2) - \psi(x)$ for all $x \in \mathbb{R}^n$. Then $\text{supp } \varphi \subset \{x \in \mathbb{R}^n : 1/2 \leq |x| < 2\} = R_0$. Let $\Phi_k(x) \equiv \varphi(2^{-k}x)$ for $x \in \mathbb{R}^n$. Then $\sum_{k \in \mathbb{Z}} \Phi_k(x) = 1$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $\text{supp } \Phi_k \subset R_k$.

Let $\{\tilde{\psi}_\beta : |\beta| \leq s\} \subset S(\mathbb{R}^n)$ being a dual basis of $\{x^\beta : |\beta| \leq s\}$ with respect to the weight $|R_0|^{-1} \varphi$, namely,

$$\frac{1}{|R_0|} \int_{\mathbb{R}^n} x^\beta \tilde{\psi}_\beta(x) \varphi(x) dx = \delta_{\beta\gamma},$$

where $\delta_{\beta\gamma} = 0$ if $\beta \neq \gamma$ and $\delta_{\beta\gamma} = 1$ if $\beta = \gamma$. By changing of the variable, we have

$$2^{-k(n+|\beta|)} \frac{1}{|R_0|} \int_{\mathbb{R}^n} x^\beta \tilde{\psi}_\beta(2^{-k}x) \varphi(2^{-k}x) dx = \delta_{\beta\gamma}.$$
For all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $|\beta| \leq s$, set

$$
\psi_{k, \beta}(x) \equiv 2^{-k(n+|\beta|)}|R_0|^{-1} \tilde{\psi}_{\beta}(2^{-k}x) \Phi_k(x).
$$

Then $\psi_{k, \beta} \in \mathcal{S}(\mathbb{R}^n)$, supp $\psi_{k, \beta} \subset R_k$.

(3.2) \[ \| \psi_{k, \beta} \|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k(n+|\beta|)} \]

and

(3.3) \[ \int_{\mathbb{R}^n} \psi_{k, \beta}(x)x^\gamma \, dx = \delta_{\beta \gamma}. \]

Let $f \in \mathcal{D}_s(\mathbb{R}^n)$. For $k \in \mathbb{Z}$, put $f_k \equiv f \Phi_k$ and

$$
P_k \equiv \sum_{|\beta| \leq s} \psi_{k, \beta} \int_{\mathbb{R}^n} f_k(y)y^\beta \, dy.
$$

It is easy to see that $f_k - P_k \in \dot{D}_s(\mathbb{R}^n)$ and supp $(f_k - P_k) \subset R_k$ for $k \in \mathbb{Z}$. Now we decompose $f$ as follows,

$$
f(x) = \sum_{k \in \mathbb{Z}} [f_k(x) - P_k(x)] + \sum_{k \in \mathbb{Z}} P_k(x)
$$

$$
= \sum_{k \in \mathbb{Z}} [f_k(x) - P_k(x)] + \sum_{k \in \mathbb{Z}} \sum_{|\beta| \leq s} [\psi_{k, \beta}(x) - \psi_{k+1, \beta}(x)] \int_{\mathbb{R}^n} \sum_{\ell = -\infty}^k f_\ell(y)y^\beta \, dy,
$$

where the equality holds for $x \in \mathbb{R}^n \setminus \{0\}$.

Notice that $|f(x)| \leq G_N(f)(x)$ for all $x \in \mathbb{R}^n$. Then by (3.2), the H"{o}lder inequality and the property of the $A_1$ weight (see [6]), we have

$$
\| P_k \|_{L^q(\mathbb{R}^n; w_2)} \leq \sum_{|\beta| \leq s} 2^{-kn} \left\{ \int_{B_{k+1}} w_2(x)^q \, dx \right\}^{1/q} \int_{\mathbb{R}^n} G_N(f)(x)\chi_{R_{k+1}}(x) \, dx
$$

$$
\leq \sum_{|\beta| \leq s} \| G_N(f)\chi_{R_k} \|_{L^q(\mathbb{R}^n; w_2)} \left\{ \frac{1}{|B_{k+1}|} \int_{B_{k+1}} [w_2(x)]^q \, dx \right\}^{1/q}
$$

$$
\times \left\{ \frac{1}{|B_{k+1}|} \int_{B_{k+1}} [w_2(x)]^{1/(q-1)} \, dx \right\}^{1-1/q} \lesssim \| \chi_{R_k} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}.
$$

This implies that

$$
\| f_k - P_k \|_{L^q(\mathbb{R}^n; w_2)} \lesssim \| \chi_{R_k} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}.
$$
Let $\lambda_k \equiv [w_1(B_{k+1})]^{\alpha/n} \| \chi_{R_k} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}$ and $a_k \equiv (\lambda_k)^{-1}(f_k - P_k)$. Then there exists a constant $C > 0$ such that $C a_k$ is a central $(\alpha, q, s; w_1, w_2)$-atom supported in $B_{k+1} \setminus B_{k-1}$. Notice that $w_1 \in A_1$ implies that there exist constant $C > 0$ and $\delta > 0$ such that $w_1(B(x, \lambda r)) \leq C \lambda^\delta w_1(B(x, r))$ for all $x \in \mathbb{R}^n$, $r > 0$ and $\lambda > 1$; see [6]. We then have

$$
\left\{ \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right\}^{1/p} \leq \left\{ \sum_{k \in \mathbb{Z}} [w_1(B_{k+1})]^{p \alpha/n} \| \chi_{R_k} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}^p \right\}^{1/p}
$$

(3.4)

To estimate the second summation, for $|\beta| \leq s$, we set $\phi((\beta,y)) \equiv \sum_{\ell = -\infty}^0 \varphi(2^{-\ell} y) \beta^\ell$ for $y \neq 0$, and $\phi((\beta,0)) = 0$ if $|\beta| > 0$, $\phi((\beta,0)) = 1$ if $|\beta| = 0$. Then there exists a constant $C > 0$ such that $C \phi((\beta)) \in S_N(\mathbb{R}^n)$ for all $|\beta| \leq s$. Moreover, it is easy to see that

$$
\left| \int_{\mathbb{R}^n} \sum_{\ell = -\infty}^k f_\ell(y)^{\beta} \, dy \right| = 2^{k(n+|\beta|)} |\phi(2^k \cdot 0)| \leq 2^{k(n+|\beta|)} \chi_{B_{k+1}}(x) G_N(f)(x).
$$

(3.5)

Notice that (3.2) and (3.3) imply that $\psi_{k,\beta} - \psi_{k+1,\beta} \in \tilde{D}_s(\mathbb{R}^n)$ and

$$
|\psi_{k,\beta} - \psi_{k+1,\beta}| \lesssim 2^{-k(n+|\beta|)} \chi_{R_k \cup R_{k+1}}.
$$

(3.6)

Let $\mu_k \equiv C[w_1(B_{k+2})]^{\alpha/n} \| \chi_{R_k} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}$ and

$$
b_k \equiv (\mu_k)^{-1} \sum_{|\beta| \leq s} (\psi_{k,\beta} - \psi_{k+1,\beta}) \int_{\mathbb{R}^n} \sum_{\ell = -\infty}^k f_\ell(y)^{\beta} \, dy.
$$

Then by (3.5) and (3.6), we have that $b_k \in \tilde{D}_s(\mathbb{R}^n)$ supported in $B_{k+2} \setminus B_{k-1}$ and $\| b_k \|_{L^q(\mathbb{R}^n; w_2)} \lesssim [w_1(B_{k+2})]^{-\alpha/n}$, which implies that there exists a constant $C > 0$ such that $C b_k \in \tilde{D}_s(\mathbb{R}^n)$ is a central $(\alpha, q, s; w_1, w_2)$-atom supported in $B_{k+2} \setminus B_{k-1}$ for all $k \in \mathbb{Z}$. By (3.5) again, we have

$$
\left\{ \sum_{k \in \mathbb{Z}} |\mu_k|^p \right\}^{1/p} \lesssim \left\{ \sum_{k \in \mathbb{Z}} [w_1(B_{k+1})]^{p \alpha/n} \| \chi_{R_k \cup R_{k+1}} G_N(f) \|_{L^q(\mathbb{R}^n; w_2)}^p \right\}^{1/p}
$$

(3.7)

Notice that if $|k - \ell| > 2$, then $\text{supp } a_k \cap \text{supp } a_\ell = \emptyset$ and $\text{supp } b_k \cap \text{supp } b_\ell = \emptyset$. Then we have $f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x) + \sum_{k \in \mathbb{Z}} \mu_k b_k(x)$ pointwise for all $x \in \mathbb{R}^n \setminus \emptyset$. This completes the proof.

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and central \((p, q, s; w_1, w_2)\) atoms gives the central atomic decomposition of \(f\).

Moreover, for \(f \in \mathcal{D}_s(\mathbb{R}^n)\), assume \(\text{supp } f \subset B_{k_0-1} \setminus B_{k_1+1}\). If \(k > k_0\) or \(k < k_1\), then \(f_k = 0\) and

\[
\int_{\mathbb{R}^n} \sum_{\ell = -\infty}^k f_k(y) y^\beta dy = \int_{\mathbb{R}^n} f(y) y^\beta dy = 0,
\]

which implies \(a_k = 0\) and \(b_k = 0\). We have a finite atomic decomposition of \(f\), i.e.,

\[
f(x) = \sum_{k=k_1}^{k_0} (\lambda_k a_k(x) + \mu_k b_k(x)),
\]

and by (3.4) and (3.7), \(\|f\|_{C_F^{\alpha,q,s}(w_1,w_2)} \lesssim \|f\|_{\dot{H}^{\alpha,p}(w_1,w_2)}\), which is desired and thus completes the proof of Theorem 1.

**Remark 1.** For any \(f \in \mathcal{D}_s(\mathbb{R}^n)\), in the proof of Theorem 1 (i), we in fact prove that \(f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x)\) pointwise for all \(x \in \mathbb{R}^n \setminus \{0\}\) and in \(\mathcal{S}'(\mathbb{R}^n)\), where \(\{a_k\}_{k \in \mathbb{Z}}\) are central \((\alpha, q, s; w_1, w_2)\)-atoms with \(\text{supp } a_k \subset B_{k+2} \setminus B_{k-1}\) in \(\mathcal{D}_s(\mathbb{R}^n)\) and \(\sum_{k \in \mathbb{Z}} |\lambda_k|^p \}^{1/p} \lesssim \|f\|_{\dot{H}^{\alpha,p}(w_1,w_2)}\). Moreover, if \(\text{supp } f \subset B_{k_0-1} \setminus B_{k_1+1}\) for some \(k_1 \in \mathbb{Z}\), then \(\lambda_k = 0\) for \(k > k_0\) and \(k < k_1\).

Based on Theorem 1 and Remark 1, we have the following conclusion.

**Lemma 1.** Let \(p \in (0,\infty), q \in (1,\infty), \alpha \in [n(1-1/q), \infty)\) and nonnegative integer \(s \geq |\alpha-n(1-1/q)|\). Then,

(i) \(C_F^{\alpha,q,s}(w_1,w_2)\) and \(\dot{F}_p^{\alpha,q,s}(w_1,w_2)\) are both dense in \(H^{\alpha,p}(w_1,w_2)\);

(ii) \(C_F^{\alpha,q,s}(w_1,w_2)\) and \(\dot{F}_p^{\alpha,q,s}(w_1,w_2)\) are both dense in \(H\dot{K}^{\alpha,p}(w_1,w_2)\).

**Proof.** Observing that \(C_F^{\alpha,q,s}(w_1,w_2) \subset \dot{F}_p^{\alpha,q,s}(w_1,w_2)\), to prove (i), we only need to prove the density of \(C_F^{\alpha,q,s}(w_1,w_2)\) in \(H^{\alpha,p}(w_1,w_2)\).

To this end, let \(f \in H\dot{K}^{\alpha,p}(w_1,w_2)\). By Theorem A (i), there exist \(\{\lambda_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}\) and central \((\alpha, q, s; w_1, w_2)\)-atoms \(\{a_k\}_{k \in \mathbb{Z}}\) with \(\text{supp } a_k \subset B_k\) such that

\[
f = \sum_{k \in \mathbb{Z}} \lambda_k a_k \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ and } \left\{\sum_{k \in \mathbb{Z}} |\lambda_k|^p \right\}^{1/p} \lesssim \|f\|_{H\dot{K}^{\alpha,p}(w_1,w_2)}.
\]

Let \(f_L = \sum_{|k| \leq L} \lambda_k a_k\) for \(L \in \mathbb{N}\). Then by Theorem A (i), \(f_L \in H\dot{K}^{\alpha,p}(w_1,w_2)\) and

\[
\|f - f_L\|_{H\dot{K}^{\alpha,p}(w_1,w_2)} = \left\|\sum_{|k| > L} \lambda_k a_k \right\|_{H\dot{K}^{\alpha,p}(w_1,w_2)} \lesssim \left\{\sum_{|k| > L} |\lambda_k|^p \right\}^{1/p} \to 0.
\]

Notice that \(f_L \in L^q(\mathbb{R}^n) \cap H\dot{K}^{\alpha,p}(w_1,w_2)\) and \(\text{supp } f \subset B_{L-1}\). Let \(\varphi \in C_0^\infty(\mathbb{R}^n)\) and \(\int_{\mathbb{R}^n} \varphi(x) dx = 1\). Then \(\varphi * f_L \in \mathcal{D}_s(\mathbb{R}^n)\). We further claim that

\[
\|\varphi * f - f_L\|_{H\dot{K}^{\alpha,p}(w_1,w_2)} \to 0 \text{ as } t \to 0.
\]

To see this, since for any \(t \in (0,2^{-L})\) and \(|k| < L\), \(|a_k|_{H\dot{K}^{\alpha,p}(w_1,w_2)} \leq 1\) and \(|w_2(B_{k+1})|^{-\alpha/2} \cdot |\varphi|_{H^{\alpha,p}(w_1,w_2)}^{\alpha/2} \to 0\) as \(t \to 0\).
Let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1-1/q), \infty)$ and nonnegative integer $s \geq \lfloor \alpha - n(1 - 1/q) \rfloor$. Then $\mathcal{D}_s(\mathbb{R}^n)$ is dense in $H_{q}^{\alpha,p}(w_1, w_2)$ and $\mathcal{D}_s(\mathbb{R}^n)$ is dense in $H_{q}^{\alpha,p}(w_1, w_2)$.

As an application of Theorem 1, we give some criteria on the boundedness of sublinear operators in $H_{q}^{\alpha,p}(w_1, w_2)$ and $H_{q}^{\alpha,p}(w_1, w_2)$.

To this end, recall that a quasi-Banach space $B$ is a vector space endowed with a quasi-norm $\| \cdot \|_B$ which is nonnegative, non-degenerate ($\| f \|_B = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality $\| f + g \|_B \leq K(\| f \|_B + \| g \|_B)$ for certain constant $K \geq 1$ and any $f, g \in B$. The notion of $p$-quasi-Banach space is given in [24].

**Definition 4.** Let $p \in (0, 1]$. A quasi-Banach spaces $B_p$ with a quasi-norm $\| \cdot \|_{B_p}$ is said to be a $p$-quasi-Banach space if $\| f + g \|_{B_p}^p \leq \| f \|_{B_p}^p + \| g \|_{B_p}^p$ for any $f, g \in B_p$.

Notice that all Banach spaces are 1-quasi-Banach spaces, and quasi-Banach spaces $L^p(\mathbb{R}^n), H^p(\mathbb{R}^n), H_{q}^{\alpha,p}(w_1, w_2), H_{q}^{\alpha,p}(w_1, w_2), H_{q}^{\alpha,p}(w_1, w_2)$ and $H_{q}^{\alpha,p}(w_1, w_2)$.
(w_1, w_2) with p \in (0, 1) are typically p-quasi-Banach spaces. Moreover, according to the Aoki-Rolewicz theorem (see [2] or [21]), any quasi-Banach space is, in essential, a p-quasi-Banach space, where p = [\log_2(2K)]^{-1}.

Recall that for any given r-quasi-Banach space B_r with r \in (0, 1] and linear space Y, an operator T from Y to B_r is called to be B_r-sublinear if for any f, g \in Y and \lambda, \nu \in \mathbb{C}, we have
\[
\|T(\lambda f + \nu g)\|_{B_r} \leq (|\lambda|^r\|T(f)\|_{B_r}^r + |\nu|^r\|T(g)\|_{B_r}^r)^{1/r}
\]
and
\[
\|T(f) - T(g)\|_{B_r} \leq \|T(f - g)\|_{B_r}.
\]

Observe that if T is linear, then T is B_r-sublinear. Moreover, if B_r = K^{\alpha,r}_{q}(w_1, w_2), K^{\alpha,r}_{q}(w_1, w_2) or L^r(\mathbb{R}^n) and T is sublinear in the classical sense, then T is also B_r-sublinear.

Another main result of this paper is as follows, which already has a lot of applications; see [11].

**Theorem 2.** Let p \in (0, 1], r \in [p, 1], q \in (1, \infty), \alpha \in [n(1 - 1/q), \infty) and nonnegative integer s \geq \lfloor \alpha - n(1 - 1/q) \rfloor.

(i) If T is a B_r-sublinear operator defined on H^{ \alpha, q, s}_{p}(w_1, w_2) such that
\[
S \equiv \sup\{\|Ta\|_{B_r} : a \text{ is any central } (\alpha, q, s; w_1, w_2)_{0}\text{-atom} \} < \infty \quad (3.8)
\]

or defined on C H^{ \alpha, q, s}_{p}(w_1, w_2) such that
\[
S \equiv \sup\{\|Ta\|_{B_r} : a \text{ is any } C_\infty(\mathbb{R}^n) \text{ central } (\alpha, q, s; w_1, w_2)_{0}\text{-atom} \} < \infty, \quad (3.9)
\]

then T uniquely extends to be a bounded B_r-sublinear operator from H K^{\alpha,p}_{q}(w_1, w_2) to B_r.

(ii) If T is a B_r-sublinear operator defined on F^{ \alpha, q, s}_{p}(w_1, w_2) such that
\[
S \equiv \sup\{\|Ta\|_{B_r} : a \text{ is any central } (\alpha, q, s; w_1, w_2) \quad (3.10)
\]

or defined on C F^{ \alpha, q, s}_{p}(w_1, w_2) such that
\[
S \equiv \sup\{\|Ta\|_{B_r} : a \text{ is any } C_\infty(\mathbb{R}^n) \text{ central } (\alpha, q, s; w_1, w_2) \quad (3.11)
\]

then T uniquely extends to be a bounded B_r-sublinear operator from H K^{\alpha,p}_{q}(w_1, w_2) to B_r.
Proof. Assume that (3.9) holds, to prove (i), let \( f \in C \hat{F}_p^{\alpha, q, s}(w_1, w_2) \). Then by Theorem 1 (i), there exist numbers \( \{ \lambda_k \}_{k=1}^m \subset \mathbb{C} \) and central \((\alpha, q, s; w_1, w_2)\)-atoms \( \{ a_k \}_{k=0}^m \subset C_c^\infty(\mathbb{R}^n) \) such that \( f = \sum_{k=1}^m \lambda_k a_k \) and \( \{ \sum_{k=1}^m | \lambda_k |^p \}^{1/p} \lesssim \| f \|_{\dot{H}^{\alpha, p}_q(w_1, w_2)} \). Since \( T \) is \( \mathcal{B}_r \)-sublinear, \( \| Ta_k \|_{\mathcal{B}_r} \lesssim 1 \) and \( r \in [p, 1] \), we have

\[
\| Tf \|_{\mathcal{B}_r} \lesssim \left\{ \sum_{k=1}^m | \lambda_k |^r \| Ta_k \|_{\mathcal{B}_r}^r \right\}^{1/r} \lesssim \left\{ \sum_{k=1}^m | \lambda_k |^p \right\}^{1/p} \lesssim \| f \|_{\dot{H}^{\alpha, p}_q(w_1, w_2)}.
\]

Then using density of \( C \hat{F}_p^{\alpha, q, s}(w_1, w_2) \) in \( \dot{H}^{\alpha, p}_q(w_1, w_2) \) given in Lemma 1 (i), we extend the \( \mathcal{B}_r \)-sublinear operator \( T \) uniquely to a bounded operator from \( \dot{H}^{\alpha, p}_q(w_1, w_2) \) to \( \mathcal{B}_r \).

If (3.8) holds, by a slight modification of the above procedure, we can also uniquely and boundedly extend \( T \) to the whole \( \dot{H}^{\alpha, p}_q(w_1, w_2) \).

The proof of (ii) is similar to that of (i). We leave the details to the reader. This completes the proof of Theorem 2.

Remark 2.

(i) If \( T \) is \( \mathcal{B}_r \)-sublinear and (3.8) holds, then (3.9) also holds automatically and thus we have two extensions of \( T \) by Theorem 2 (i). Since both of the two extensions are unique and coincide on \( \hat{F}_p^{\alpha, q, s}(w_1, w_2) \), by Lemma 1 (i), the two extensions of \( T \) coincide on \( \dot{H}^{\alpha, p}_q(w_1, w_2) \). Similarly, if (3.10) holds, then we have two extensions of \( T \) which coincide.

(ii) The conditions (3.8) or (3.9) and (3.10) or (3.11) are also necessary. Moreover, even when \( \mathcal{B}_r \equiv \dot{K}^{\alpha, p}_q(w_1, w_2) \) (resp. \( \mathcal{B}_r \equiv K^{\alpha, p}_q(w_1, w_2) \)), Theorem 2 also makes an improvement of Theorem 2 with \( p \in (0, 1] \) in [14] by removing the \( L^q(\mathbb{R}^n; w_2) \)-boundedness of \( T \) and some size conditions therein. In fact, in Theorem 2, we do not need the \( L^q(\mathbb{R}^n; w_2) \)-boundedness of \( T \) or the continuity of \( T \) from \( \mathcal{S}(\mathbb{R}^n) \) to \( \mathcal{S}'(\mathbb{R}^n) \). This is convenient in applications of Herz-type Hardy spaces to the boundedness of sublinear operators.

(iii) Theorem 2 with \( p \in (1, \infty) \) in [14] still holds by using Lemma 1 and Theorem 1 to seal a gap in the proof in [14]. We leave the details to the reader.

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References


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