MULTIPLE COMBINATORIAL STOKES’ THEOREM WITH BALANCED STRUCTURE

Shyh-Nan Lee, Chien-Hung Chen and Mau-Hsiang Shih*

Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. Combinatorics of complexes plays an important role in topology, nonlinear analysis, game theory, and mathematical economics. In 1967, Ky Fan used door-to-door principle to prove a combinatorial Stokes’ theorem on pseudomanifolds. In 1993, Shih and Lee developed the geometric context of general position maps, \(\pi\)-balanced and \(\pi\)-subbalanced sets and used them to prove a combinatorial formula for multiple set-valued labellings on simplexes. On the other hand, in 1998, Lee and Shih proved a multiple combinatorial Stokes’ theorem, generalizing the Ky Fan combinatorial formula to multiple labellings. That raises a question: Does there exist a unified theorem underlying Ky Fan’s theorem and Shih and Lee’s results? In this paper, we prove a multiple combinatorial Stokes’ theorem with balanced structure. Our method of proof is based on an incidence function. As a consequence, we obtain a multiple combinatorial Sperner’s lemma with balanced structure.

1. INTRODUCTION

The theory of combinatorics of complexes may be traced back to 1928 [14] when Sperner discovered a combinatorial lemma, that is globally called Sperner’s lemma, which gave a drastic simplification of proofs of two topological theorems, namely theorems of invariance of domain and invariance of dimension. In 1929, Knaster, Kuratowski and Mazurkiewicz [5] used Sperner’s lemma to give a combinatorial proof of Brouwer’s fixed-point theorem. In 1967, Scarf [10] used Sperner’s lemma to give a constructive proof of Brouwer’s fixed-point theorem and in 1974, Kuhn [6] gave a constructive proof of the fundamental theorem of algebra based on the

The purpose of this paper is to give further generalizations of Lee and Shih’s results concerning Stokes’ theorem on pseudomanifolds [13]. We prove a multiple combinatorial Stokes’ theorem with balanced structure. Instead of a search algorithm modifying Ky Fan’s door-to-door principle [2], we prove our result by an incidence function. As a consequence, we obtain a multiple combinatorial Sperner’s lemma with balanced structure. The paper is organized as follows. In Section 2 we introduce some basic definitions and notations. In Section 3 we study some properties of general position maps, $\pi$-balanced and $\pi$-subbalanced collections. In Sections 4 and 5 we prove our main results, multiple combinatorial Stokes’ theorem with balanced structure and multiple combinatorial Sperner’s lemma with balanced structure.

2. Definitions and Notations

For convenience sake, we recall some definitions and notations in this section. The notion of pseudomanifolds is an abstraction of surfaces and curves in a discrete sense which may be defined as follows, see also [2].

An (abstract) complex is a finite collection $\mathcal{K}$ of nonempty finite sets such that

(K1) if $\sigma \in \mathcal{K}$ and $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{K}$.

Elements of $\mathcal{K}$ are called simplexes of $\mathcal{K}$. A simplex $\sigma$ of $\mathcal{K}$ is called a $k$-simplex of $\mathcal{K}$ if the cardinality $|\sigma|$ of $\sigma$ is $k+1$. For a $k$-simplex $\sigma$ of $\mathcal{K}$, the subsets $\tau$ of
\[ \sigma \] such that \(|\tau| = r + 1\) are called the \(r\)-faces of \(\sigma\) \((0 \leq r \leq k)\). The vertex set \(V(\mathcal{K})\) of \(\mathcal{K}\) is the union of all simplexes of \(\mathcal{K}\).

Let \(n\) be a positive integer. An \((n - 1)\)-pseudomanifold is a complex \(\mathcal{K}\) with the following two properties:

(M1) Every simplex of \(\mathcal{K}\) is a face of at least one \((n - 1)\)-simplex of \(\mathcal{K}\).

(M2) Every \((n - 2)\)-simplex of \(\mathcal{K}\) is a common face of at most two distinct \((n - 1)\)-simplexes of \(\mathcal{K}\).

An \((n - 2)\)-simplex \(\tau\) of an \((n - 1)\)-pseudomanifold \(\mathcal{K}\) is called a boundary \((n - 2)\)-simplex of \(\mathcal{K}\) if \(\tau\) is a face of exactly one \((n - 1)\)-simplex of \(\mathcal{K}\). The set of all boundary \((n - 2)\)-simplexes and their faces is denoted by \(\partial \mathcal{K}\).

Sometimes oriented simplexes, that is, simplexes with orientations, are considered. The notion of the orientations of a nonempty finite set is defined below.

Let \(\sigma = \{v_1, \ldots, v_n\}\) be a finite set of cardinality \(n \geq 2\). We call an \(n\)-tuple with distinct components of the elements of \(\sigma\) an ordering of \(\sigma\). Two orderings \((v_{i_1}, \ldots, v_{i_n})\) and \((v_{j_1}, \ldots, v_{j_n})\) of \(\sigma\) are said to have the same orientation if the permutation \((v_{i_1}, \ldots, v_{i_n})\) is even. Having the same orientation is an equivalence relation and it partitions the set of \(n!\) orderings of \(\sigma\) into two equivalence classes. Each of the equivalence classes is called an orientation on \(\sigma\), and if we fix one of them arbitrarily, the other one is called the opposite orientation. The orientation on \(\sigma\) determined by the ordering \((v_{i_1}, \ldots, v_{i_n})\) is denoted by \((+1)[v_{i_1}, \ldots, v_{i_n}]\) and the opposite orientation of \((+1)[v_{i_1}, \ldots, v_{i_n}]\) is denoted by \((-1)[v_{i_1}, \ldots, v_{i_n}]\). For the case \(n = 1\), we call the two symbols \((+1)[v_1]\) and \((-1)[v_1]\) orientations on the one-point set \(\{v_1\}\) and they are defined to be opposite orientations on \(\{v_1\}\).

Given an orientation \(\omega = \varepsilon[v_1, \ldots, v_n]\) on the set \(\sigma = \{v_1, \ldots, v_n\}\) where \(\varepsilon = \pm 1\) and \(n \geq 2\). For each \(i = 1, \ldots, n\), the induced orientation on \(\sigma \setminus \{v_i\}\) from \(\omega\) is the well defined orientation \((-1)^{i-1}\varepsilon[v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]\) on \(\sigma \setminus \{v_i\}\).

We now consider the notion of orientable pseudomanifolds which is an abstraction of orientable surfaces in a discrete sense.

Let \(\mathcal{K}\) be an \((n - 1)\)-pseudomanifold. If there is a map \(\omega\) defined on the set of all \((n - 1)\)-simplexes of \(\mathcal{K}\) such that following two conditions are satisfied:

(C1) For each \((n - 1)\)-simplex \(\sigma\) of \(\mathcal{K}\), \(\omega(\sigma)\) is an orientation on \(\sigma\).

(C2) If \(\tau\) is an \((n - 2)\)-simplex of \(\mathcal{K}\) which is a common face of two distinct \((n - 1)\)-simplexes \(\sigma_1\) and \(\sigma_2\) of \(\mathcal{K}\), then \(\omega(\sigma_1)\) and \(\omega(\sigma_2)\) induce opposite orientations on \(\tau\).

Then \(\mathcal{K}\) is said to be orientable and the ordered pair \((\mathcal{K}, \omega)\) is called a coherently oriented \((n - 1)\)-pseudomanifold.

The following notion of triangulations of geometric simplexes are also needed in this paper.

Given a finite set \(\sigma = \{v_1, \ldots, v_n\}\) in a Euclidean space, the affine hull \(aff(\sigma)\) of \(\sigma\) is the set of all affine combinations of \(v_1, \ldots, v_n\), that is, \(aff(\sigma) = \)
\[ \sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1 \], and the convex hull \( \text{conv}(\sigma) \) of \( \sigma \) is the set of all convex combinations of \( v_1, \ldots, v_n \), that is, \( \text{conv}(\sigma) = \left\{ \sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1, \text{ each } \lambda_i \geq 0 \right\} \).

The set \( \sigma \) is said to be affinely independent if the following is true:

(A1) If \( \sum_{i=1}^{n} \lambda_i v_i = 0 \) and if \( \sum_{i=1}^{n} \lambda_i = 0 \) then \( \lambda_1 = \ldots = \lambda_n = 0 \).

If \( \sigma = \{v_1, \ldots, v_n\} n \geq 1 \) is affinely independent, the convex hull of \( \sigma \) is called the \((n-1)\)-simplex spanned by \( \sigma \), the points \( v_1, \ldots, v_n \) are called the vertices of the simplex, and we denote this simplex by \( \overrightarrow{v_1 \ldots v_n} \). For \( 1 \leq i_1 < \ldots < i_k \leq n \), the \((k-1)\)-simplex \( \overrightarrow{v_{i_1} \ldots v_{i_k}} \) is called a \((k-1)\)-face of the \((n-1)\)-simplex \( \overrightarrow{v_1 \ldots v_n} \). The open simplex spanned by the set \( \sigma \) is the set \( \text{Int}(\overrightarrow{v_1 \ldots v_n}) = \left\{ \sum_{i=1}^{n} \lambda_i v_i; \sum_{i=1}^{n} \lambda_i = 1, \text{ each } \lambda_i > 0 \right\} \).

A finite collection \( T \) of (geometric) simplexes is called a triangulation of an \((n-1)\)-simplex \( \overrightarrow{a_1 \ldots a_n} \) if the following three conditions are satisfied:

(T1) \( \overrightarrow{a_1 \ldots a_n} = \bigcup_{s \in T} s \).

(T2) If \( s \in T \) and if \( t \) is a face of \( s \) then \( t \in T \).

(T3) If \( s, t \in T \) and \( s \cap t \neq \emptyset \), then \( s \cap t \) is a common face of \( s \) and \( t \).

A point \( v \in \overrightarrow{a_1 \ldots a_n} \) is called a vertex of \( T \) if \( v \) is a vertex of some simplex of \( T \). The set of all vertices of \( T \) is denoted by \( V(T) \). Let \( \hat{T} \) be the collection of all affinely independent subsets \( \sigma \) of \( V(T) \) that spans a simplex of \( T \). Then \( \hat{T} \) is an (abstract) complex, known as the vertex scheme of \( T \). Let us now fix an orientation \( \omega = \varepsilon[a_1, \ldots, a_n] \) \((\varepsilon = \pm 1)\) on the set \( \{a_1, \ldots, a_n\} \). For each \((n-1)\)-simplex \( \sigma = \{v_1, \ldots, v_n\} \) of \( \hat{T} \), define an orientation \( \omega(\sigma) \) on \( \sigma \) by \( \omega(\sigma) = \varepsilon[v_1, \ldots, v_n] \) or \( \omega(\sigma) = (-1)\varepsilon[v_1, \ldots, v_n] \) according as \( \det(\alpha_{ij}) > 0 \) or \( \det(\alpha_{ij}) < 0 \) respectively, where \( (\alpha_{ij}) \) is the \( n \times n \) real matrix satisfying

\[
(2.1) \quad v_i = \sum_{j=1}^{n} \alpha_{ij} a_j \quad (\sum_{j=1}^{n} \alpha_{ij} = 1) \quad \text{for } i = 1, \ldots, n.
\]

Thus we may view an orientation \( \omega \) on \( \{a_1, \ldots, a_n\} \) a map whose domain is the set of all \((n-1)\)-simplexes of \( T \) which assigns an orientation \( \omega(\sigma) \) to \( \sigma \) for each \((n-1)\)-simplex \( \sigma \) of \( \hat{T} \). It is well known that \((\hat{T}, \omega)\) forms a coherently oriented \((n-1)\)-pseudo-manifold.

The following geometric context of general position maps, \( \pi \)-balanced and \( \pi \)-sub-balanced collections compare with [11], are given in [13] and they are the key definitions of our theory.
For a given nonempty finite set $N$, the collection of all nonempty subsets of $N$ is denoted by $2^N$. Thus $|2^N| = 2^{|N|} - 1$. Let $A = \{a_i\}_{i \in N}$ be an affinely independent set. If $S \in 2^N$, the simplex spanned by $\{a_i\}_{i \in S}$ is denoted by $A^S$, and the barycenter of $A^S$ is the point $m_S = \frac{1}{|S|} \sum_{i \in S} a_i$. Let $\pi : 2^N \rightarrow A^N$ and $p \in P \in 2^N$. We say that $\pi$ is a Shapley map if

(S1) for each $S \in 2^N, \pi(S) \in A^S$,

and that $\pi$ is a general position map if the following two conditions are satisfied:

(G1) For each $S \in 2^N, \pi(S) \in \text{Int}(A^S)$.

(G2) For each $S \in 2^N, f$ is $\pi$-balanced with respect to $P$ if it satisfies the following two conditions:

(B1) $B \subset 2^P$.

(B2) $m_P \in \text{conv}(\pi(B))$.

A collection $B$ of subsets of $N$ is said to be $\pi$-subbalanced with respect to $(P, p)$ if it satisfies the following two conditions:

(SB1) $B \subset 2^P$.

(SB2) $\text{conv}(\pi(B)) \cap (m_P, m_{P\setminus\{p\}}) \neq \emptyset$

where $(m_P, m_{P\setminus\{p\}}) = \{(1 - \lambda)m_P + \lambda m_{P\setminus\{p\}}; 0 < \lambda \leq 1\}$.

To formulate the multiple combinatorial Stokes’ theorem and Sperner’s lemma with balanced structure, we need to introduce the following further notations.

Let $m$ and $n$ be positive integers, let $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$, let $\pi : 2^N \rightarrow A^N$ where $A^N = a_1 \ldots a_m$, and let $K$ be an $(n-1)$-pseudomanifold. An $m$-labelling in $K$ is a multiple set-valued map $\varphi : V(K) \rightarrow (2^N)^m$ where $(2^N)^m$ is the Cartesian product $2^N \times \ldots \times 2^N$ of $m$ factors. For each vertex $v$ of $K$, $\varphi(v)$ is an $m$-tuple $(\varphi_1(v), \ldots, \varphi_m(v))$ where each $\varphi_i(v) \in 2^N$. Given $\sigma \in K$ and $f : \sigma \rightarrow M$, we shall use the notation $\varphi_f(\sigma)$ to denote the subcollection $\{\varphi_{f(v)}(v); v \in \sigma\}$ of $2^N$. Let $p \in P \in 2^N$. If the collection $\varphi_f(\sigma)$ is $\pi$-balanced with respect to $P$ or $\pi$-subbalanced with respect to $(P, p)$, then we call the pair $(\sigma, f)$ a $\pi$-balanced pair with respect to $P$ or a $\pi$-subbalanced pair with respect to $(P, p)$. The pair $(\sigma, f)$ is called a boundary pair if $\sigma \in \partial K$. The set of all $\pi$-balanced pairs with respect to $N$ is denoted by $K_\pi(\varphi)$ and the set of all $\pi$-subbalanced boundary pairs with respect to $(N, n)$ is denoted by $\partial K_\pi(\varphi)$. Suppose further that $K$ is orientable. Let $(K, \omega)$ be a coherently oriented $(n-1)$-pseudomanifold and let $\omega' = \varepsilon'[a_1, \ldots, a_n] (\varepsilon' = \pm 1)$ be an orientation on $A = \{a_1, \ldots, a_n\}$. We define
the sets \( K^+_{\pi}(\varphi) \), \( K^-_{\pi}(\varphi) \), \( \partial K^+_{\pi}(\varphi) \), and \( \partial K^-_{\pi}(\varphi) \) as follows. For a given \((n-1)\)-simplex \( \sigma = \{v_1, \ldots, v_n\} \) and a map \( f : \sigma \to M \). Let \( \omega(\sigma) = \varepsilon[v_1, \ldots, v_n] \) \((\varepsilon = \pm 1)\) and let

\[
\pi(\varphi_f(v_i)(v_i)) = \sum_{j=1}^{n} \beta_{ij}a_j \quad (\sum_{j=1}^{n} \beta_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, n.
\]

We call the pair \((\sigma, f)\) a positive pair or a negative pair if \( \varepsilon' detB > 0 \) or \( \varepsilon' detB < 0 \) respectively, where \( B \) is the \( n \times n \) matrix \((\beta_{ij})\). The sets of all positive and negative pairs of \( K_{\pi}(\varphi) \) are denoted by \( K^+_{\pi}(\varphi) \) and \( K^-_{\pi}(\varphi) \) respectively. For a given boundary \((n-2)\)-simplex \( \tau = \{v_1, \ldots, v_{n-1}\} \) and a map \( g : \tau \to M \). Let \( \sigma \) be the unique \((n-1)\)-simplex of \( K \) containing \( \tau \), \( \omega(\sigma) \) induce the orientation \( \varepsilon[v_1, \ldots, v_{n-1}] \) \((\varepsilon = \pm 1)\) on \( \tau \) and let

\[
\pi(\varphi_g(v_i)(v_i)) = \sum_{j=1}^{n} \gamma_{ij}a_j \quad (\sum_{j=1}^{n} \gamma_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, n-1.
\]

We call the pair \((\tau, g)\) a positively boundary pair or a negatively boundary pair if \( \varepsilon' detC > 0 \) or \( \varepsilon' detC < 0 \) respectively, where \( C \) is the \( (n-1) \times (n-1) \) matrix of the first \( n-1 \) columns of the \( (n-1) \times n \) matrix \((\gamma_{ij})\). The sets of all positively and negatively boundary pairs of \( \partial K_{\pi}(\varphi) \) are denoted by \( \partial K^+_{\pi}(\varphi) \) and \( \partial K^-_{\pi}(\varphi) \) respectively. Given any set \( \Omega \) of pairs \((\sigma, f)\) where \( \sigma \in K \) and \( f : \sigma \to M \), the set of all pairs \((\sigma, f)\) of \( \Omega \) such that \( f \) is one-to-one is denoted by \( \Omega_\ast \). For example, we have \( \partial K^-_{\pi}(\varphi)_\ast = \{(\sigma, f) \in \partial K^-_{\pi}(\varphi) ; f \text{ is one-to-one}\} \).

Let \( T \) be a triangulation of an \((n-1)\)-simplex \( A^N = \overrightarrow{a_1} \ldots \overrightarrow{a_n} \), let \( \varphi : V(T) \to (2^N)^m \) and let \( \pi : 2^N \to A^N \), where \( m \) and \( n \) are positive integers, \( N = \{1, \ldots, n\} \), and where \( A = \{\overrightarrow{a_1}, \ldots, \overrightarrow{a_n}\} \) is an affinely independent set. Let \( \overrightarrow{T} \) be the vertex scheme of \( T \). Then \( \varphi \) is an \( m \)-labelling in the \((n-1)\)-pseudomanifold \( \overrightarrow{T} \). Let \( M = \{1, \ldots, m\} \). Given a \((k-1)\)-simplex \( s \) of \( T \) and a map \( f : \sigma \to M \) where \( \sigma \) is the vertex set of \( s \). The pair \((s, f)\) is said to be \( k \)-labelled under \((\varphi, \pi)\) if

\[(L1)\] there exists a \( P \in 2^N \) such that \(|P| = k, s \subset A^P, \) and \( \varphi_f(\sigma) \) is \( \pi \)-balanced with respect to \( P \).

The pair \((s, f)\) is said to be fixed under \((\varphi, \pi)\) if

\[(F1)\] \( \pi(\varphi_f(\sigma)) \subset aff(f(\sigma)) \).

Let \((s, f)\) be a fixed pair under \((\varphi, \pi)\), let \( s = \overrightarrow{v_1} \ldots \overrightarrow{v_k} \), and let

\[
\pi(\varphi_f(v_i)(v_i)) = \sum_{j=1}^{k} \lambda_{ij}v_j \quad (\sum_{j=1}^{k} \lambda_{ij} = 1) \quad \text{for} \quad i = 1, \ldots, k.
\]

We call the pair \((s, f)\) a positively fixed pair or a negatively fixed pair under \((\varphi, \pi)\) if \( det(\lambda_{ij}) > 0 \) or \( det(\lambda_{ij}) < 0 \) respectively. For each \( P \in 2^N \), the set of all pairs
(s, f) such that s ⊂ A P and f : σ → M, where s ∈ T and σ is the vertex set of s, is denoted by H P. We define H s P to be the set of all pairs (s, f) ∈ H P such that f is one-to-one. Then the number of positively fixed k-labelled pairs minus the number of negatively fixed k-labelled pairs, in H P or H s P, under (φ, π) is denoted by φ k P or φ k s P, respectively.

3. BALANCEDNESS AND GENERAL POSITION MAPS

We shall list some basic properties of general position maps, π-balanced and π-subbalanced collections here, for their detail proofs please see [13].

It follows from (G1) that

(II1) general position maps are one-to-one.

(B1), (B2), (SB1) and (SB2) give that

(II2) if B ⊂ 2 N is π-subbalanced with respect to (P, p) then B ∪ { {p} } is π-balanced with respect to P.

Let π0 : 2 N → A N be a Shapley map. From (S1), (G1) and (G2), it follows that

(II3) for each ε > 0 there exists a general position map π : 2 N → A N such that max S∈2 N ∥π(S) − π0(S)∥ < ε where ∥·∥ is the Euclidean norm;

when ε > 0 is taken small enough, we have the following additional property:

(II4) if P ∈ 2 N and if B ⊂ 2 N is π-balanced with respect to P, then B is π0-balanced with respect to P.

It follows from (G1), (G2), (B1), (B2), (SB1), (SB2) and Carathéodory theorem that the following (II5) ~ (II8) are always true under the condition that π : 2 N → A N is a general position map.

(II5) A minimal π-balanced collection B ⊂ 2 N with respect to P ∈ 2 N is of cardinality |P| and the set π(B) spans a (|P| − 1)-simplex, moreover, the barycenter m P of A P is contained in the open simplex Int(conv(π(B))).

(II6) A minimal π-subbalanced collection B ⊂ 2 N with respect to (P, p) (p ∈ P ∈ 2 N with |P| ≥ 2) is of cardinality |P| − 1 and the set π(B) spans a (|P| − 2)-simplex, moreover, the set (m P, m P \ {p}) ∩ conv(π(B)) is a singleton which is contained in the open simplex Int(conv(π(B))).

(II7) If p ∈ P ∈ 2 N (|P| ≥ 2), if B ⊂ 2 N is π-balanced with respect to P, and if |B| = |P|, then there is a unique subcollection B 1 of B such that B 1 is π-subbalanced with respect to (P, p), and |B 1| = |P| − 1.
(Π8) If \( p \in P \subseteq 2^N (|P| \geq 2) \), if \( B \subset 2^N \) is \( \pi \)-subbalanced with respect to \((P, p)\) but is not \( \pi \)-balanced with respect to \( P \), and if \(|B| = |P|\), then there are exactly two subcollections \( B_1 \) and \( B_2 \) of \( B \) such that they are \( \pi \)-subbalanced with respect to \((P, p)\), and \(|B_1| = |B_2| = |P| - 1\).

We conclude this section by proving the following (Π9) and (Π10).

Let \( N = \{1, \ldots, n\} \ (n \geq 2) \), \( A = \{a_i\}_{i \in N} \) an affinely independent set, \( \pi : 2^N \rightarrow A^N \) a general position map, \( B = \{S_1, \ldots, S_n\} \subset 2^N \), and

\[
\pi(S_i) = \sum_{j=1}^{n} \alpha_{ij} a_j \quad (\sum_{j=1}^{n} \alpha_{ij} = 1) \quad \text{for} \ i = 1, \ldots, n.
\]

(Π9) If \( B \) is \( \pi \)-balanced with respect to \( N \) and \( B_1 = \{S_1, \ldots, S_{n-1}\} \) is \( \pi \)-subbalanced with respect to \((N, n)\) then

\[
(detB)(detB_1) > 0
\]

where \( B \) is the \( n \times n \) matrix \((\alpha_{ij})\) and where \( B_1 \) is the \((n - 1) \times (n - 1)\) matrix obtained by deleting the \( n \)th row and the \( n \)th column of \((\alpha_{ij})\).

(Π10) If \( B_1 = B \setminus \{S_n\} \) and \( B_2 = B \setminus \{S_{n-1}\} \) are \( \pi \)-subbalanced with respect to \((N, n)\) then

\[
(detB_1)(detB_2) > 0
\]

where \( B_1 \) is the same in (3.2) and \( B_2 \) is the \((n - 1) \times (n - 1)\) matrix obtained by deleting the \((n - 1)\)th row and the \( n \)th column of \((\alpha_{ij})\).

We now begin to prove (Π9) and (Π10). Let \( E = \{e_1, \ldots, e_n\} \) be the standard basis of the Euclidean \( n \)-space \( \mathbb{R}^n \). The affine map \( F : aff(A) \rightarrow aff(E) \) defined by \( F(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} \lambda_i e_i \ (\sum_{i=1}^{n} \lambda_i = 1) \) preserves affine combinations and affinely independent sets, moreover, since the origin of \( \mathbb{R}^n \) is not contained in \( aff(E) \), we have

(Π11) a subset of \( aff(E) \) is affinely independent if and only if it is linearly independent.

Thus, by taking an affine map if necessary, we may assume that \( A = E \) and \( a_1 = e_1, \ldots, a_n = e_n \). It follows that

\[
detB = det(\pi(S_1), \ldots, \pi(S_n)),
\]

\[
detB_1 = det(\pi(S_1), \ldots, \pi(S_{n-1}), a_n), \text{ and}
\]

\[
detB_2 = det(\pi(S_1), \ldots, \pi(S_{n-2}), \pi(S_n), a_n).
\]
Here, $\det(\pi(S_1), \ldots, \pi(S_n))$ means that if we express $\pi(S_1), \ldots, \pi(S_n)$ in terms of $e_1, \ldots, e_n$, that is, by (3.1), $\pi(S_i) = \sum_{j=1}^{n} \beta_{ij} e_j = (\beta_{i1}, \ldots, \beta_{in})$ ($i = 1, \ldots, n$), then $\det(\pi(S_1), \ldots, \pi(S_n)) = \det(\beta_{ij})$, and so on.

To prove (3.2), let us assume that $B$ is $\pi$-balanced with respect to $N$ and $B_1$ is $\pi$-subbalanced with respect to $(N, n)$. By (II5) and (II1), $\pi_B$ is linearly independent, so that

(3.7) $\det(\pi(S_1), \ldots, \pi(S_n)) \neq 0$.

(II5) also shows that $m_N \in \text{Int}(\text{conv}(\pi_B))$, so that

(3.8) $m_N = \sum_{i=1}^{n} \lambda_i \pi(S_i)$ where $\sum_{i=1}^{n} \lambda_i = 1$ and each $\lambda_i > 0$.

Similarly, by (II6), we have

(3.9) $\alpha_1 m_{N \setminus \{n\}} + (1 - \alpha_1) m_N = \sum_{i=1}^{n-1} \mu_i \pi(S_i)$ where $0 < \alpha_1 \leq 1$ and each $\mu_i > 0$.

From (3.8) and (3.9), it follows that

(3.10) $a_n = \sum_{i=1}^{n-1} \left( \lambda_i (1 + \frac{n - 1}{\alpha_1}) - \mu_i (n - 1) \right) \pi(S_i) + \lambda_n (1 + \frac{n - 1}{\alpha_1}) \pi(S_n)$.

By (3.4), (3.5) and (3.10) we have

(3.11) $\det B_1 = \lambda_n (1 + \frac{n - 1}{\alpha_1}) \det B$.

From (3.4), (3.7) and (3.11), (3.2) follows.

To see (3.3), let $B_1$ and $B_2$ be $\pi$-subbalanced with respect to $(N, n)$. By (II2), the collection $B_1 \cup \{\{n\}\}$ is $\pi$-balanced with respect to $N$, if we replace $B$ in (II9) by $B_1 \cup \{\{n\}\}$, then (3.2) implies that

(3.12) $\det B_1 \neq 0$.

From (II6), it follows that (3.9) and the following (3.13) hold:

(3.13) $\alpha_2 m_{N \setminus \{n\}} + (1 - \alpha_2) m_N$

$= \sum_{i=1}^{n-2} \mu_i' \pi(S_i) + \mu_n' \pi(S_n)$ where $0 < \alpha_2 \leq 1$ and $\mu_n' > 0$.
By substituting
\[ m_N = \frac{n-1}{n}m_{N\setminus\{n\}} + \frac{1}{n}a_n \]
into the above (3.9) and (3.13), we obtain two equations without the term of \( m_N \).
If we use these two equations to eliminating \( m_{N\setminus\{n\}} \), then we have
\[
\delta_2 \mu'_n \pi(S_n) = \sum_{i=1}^{n-2} \left( \delta_1 \mu_i - \delta_2 \mu'_i \right) \pi(S_i) + \delta_1 \mu_{n-1} \pi(S_{n-1})
\]
\[
\delta_2 \mu'_n \pi(S_n) + \left\{ \frac{\delta_2 (1-\alpha_2)}{n} - \frac{\delta_1 (1-\alpha_1)}{n} \right\} a_n
\]
where
\[
\delta_i = \{ \alpha_i + (1-\alpha_i)\frac{n-1}{n} \}^{-1} \quad (i = 1, 2).
\]
(3.5), (3.6) and (3.14) imply that
\[
\text{det}B_2 = \frac{\delta_1 \mu_{n-1}}{\delta_2 \mu'_n} \text{det}B_1.
\]
As \( \delta_1, \delta_2, \mu_{n-1} \) and \( \mu'_n \) are positive, (3.3) follows from (3.12) and (3.15).

4. MULTIPLE COMBINATORIAL STOKES’ THEOREM WITH BALANCED STRUCTURE

In 1974, Kuhn [6] gave a constructive proof of the fundamental theorem of algebra based on the combinatorial Stokes’ Theorem which is a generalization of the celebrated Sperner’s lemma [14]. Under the considerations of multiple labellings and balanced structures, Shih and Lee [13] established a combinatorial formula as a generalized Sperner’s lemma that is a unification of the results of Shapley [11](balanced version) and Bapat [1](multiple version). The following multiple combinatorial Stokes’ theorem with balanced structure generalizes the above formula.

**Theorem 1.** Let \( \varphi \) be an \( m \)-labelling in an \( (n-1) \)-pseudomanifold \( \mathcal{K} \) where \( m \) and \( n \) are positive integers with \( n \geq 2 \), and let \( \pi : 2^N \rightarrow A^N \) be a general position map where \( N = \{1, \ldots, n\} \) and \( A = \{a_1, \ldots, a_n\} \) is an affinely independent set. Then
\[
|\mathcal{K}_\pi(\varphi)| \equiv m|\partial \mathcal{K}_\pi(\varphi)| \quad (\text{mod} \ 2)
\]
and
\[
|\mathcal{K}_\pi(\varphi)_*| \equiv (m - n + 1)|\partial \mathcal{K}_\pi(\varphi)_*| \quad (\text{mod} \ 2).
\]
Suppose further, \((K, \omega)\) is a coherently oriented \((n - 1)\)-pseudomanifold, then
\[
(4.3) \quad (-1)^{n-1} \{|K^+_\pi(\phi)| - |K^-_\pi(\phi)|\} = m\{|\partial K^+_\pi(\varphi)| - |\partial K^-_\pi(\varphi)|\},
\]
and
\[
(4.4) \quad (-1)^{n-1} \{|K^+_\pi(\varphi)_*| - |K^-_\pi(\varphi)_*|\} = (m - n + 1)\{|\partial K^+_\pi(\varphi)_*| - |\partial K^-_\pi(\varphi)_*|\},
\]
that is, (4.3) and (4.4) are independent of the choices of the orientations \(\varepsilon' [\epsilon_1, \ldots, \epsilon_m] (\varepsilon' = \pm 1)\) on \(A\).

The idea of the following proof of Theorem 1 is by counting the sums of incidences between two sets \(D\) and \(R\) in two different ways.

Let \(M = \{1, \ldots, m\}\) as before. Let
\[
(5.5) \quad D = \{(\tau, g) \mid \tau \in K, \; |\tau| = n - 1, \; g : \tau \to M\}
\]
and
\[
(5.6) \quad R = \{(\sigma, f) \mid \sigma \in K, \; |\sigma| = n, \; f : \sigma \to M\}.
\]
By our definition as above, we have
\[
(5.7) \quad D^* = \{(\tau, g) \in D \mid g \text{ is one-to-one} \}
\]
and
\[
(5.8) \quad R^* = \{(\sigma, f) \in R \mid f \text{ is one-to-one} \}.
\]
Define an incidence relation \(\prec\) from \(D\) into \(R\) by \((\tau, g) \prec (\sigma, f)\) if and only if the following three conditions are satisfied:

\((R1)\) \((\tau, g) \in D\) and \((\sigma, f) \in R\),

\((R2)\) \(\varphi_g(\tau)\) is \(\pi\)-subbalanced with respect to \((N, n)\),

\((R3)\) \(\tau \subset \sigma\) and \(g = f |_{\tau}\) (the restriction of \(f\) to \(\tau\)).

Put
\[
D_1 = \{(\tau, g) \in D \mid \tau \in \partial K \text{ and } \varphi_g(\tau) \text{ is } \pi\text{-subbalanced with respect to } (N, n)\},
\]
\[
D_2 = \{(\tau, g) \in D \mid \tau \notin \partial K \text{ and } \varphi_g(\tau) \text{ is } \pi\text{-subbalanced with respect to } (N, n)\},
\]
\[
D_3 = \{(\tau, g) \in D \mid \varphi_g(\tau) \text{ is not } \pi\text{-subbalanced with respect to } (N, n)\},
\]
\[
R_1 = \{(\sigma, f) \in R \mid \varphi_f(\sigma) \text{ is } \pi\text{-balanced with respect to } N\},
\]
\[
R_2 = \{(\sigma, f) \in R \mid \varphi_f(\sigma) \text{ is } \pi\text{-subbalanced with respect to } (N, n)\}
but not \pi\text{-balanced with respect to } N\},
\]
\[
R_3 = \{(\sigma, f) \in R \mid \varphi_f(\sigma) \text{ is not } \pi\text{-subbalanced with respect to } (N, n)\}.
\]
It is clear that \[ \{D_1, D_2, D_3\} \text{ partitions } D \]
and \[ \{D_{1*}, D_{2*}, D_{3*}\} \text{ partitions } D_* \].

From (II1), (II7) and (4.6), it follows that if \( \varphi_f(\sigma) \) is \( \pi \)-balanced with respect to \( N \) then it is \( \pi \)-subbalanced with respect to \( (N, n) \), so that \[ \{R_1, R_2, R_3\} \text{ partitions } R \]
and \[ \{R_{1*}, R_{2*}, R_{3*}\} \text{ partitions } R_* \].

Let \[ D_r = \{d \in D \mid d < r\} \ (r \in R) \]
and \[ R_d = \{r \in R \mid d < r\} \ (d \in D) \].

We claim that \( |D_r| = 1 \ (r \in R_1) \), \( |D_r| = 2 \ (r \in R_2) \), \( |D_r| = 0 \ (r \in R_3) \),
and \( |D_{r*}| = 1 \ (r \in R_{1*}) \), \( |D_{r*}| = 2 \ (r \in R_{2*}) \), \( |D_{r*}| = 0 \ (r \in R_{3*}) \).

To see (4.11) \( \sim \) (4.13), let us fix \( r = (\sigma, f) \in R \). By (4.6), we have \[ |\varphi_f(\sigma)| = |\{\varphi_f(v) \mid v \in \sigma\}| \leq |\sigma| = n. \]

Let \[ \sigma = \{v_1, \ldots, v_n\}, \ \tau_1 = \sigma \setminus \{v_n\}, \ \tau_2 = \sigma \setminus \{v_{n-1}\}, \]
\[ g_1 = f \mid \tau_1 \text{ and } g_2 = f \mid \tau_2. \]
Then, by (K1), \( \tau_1 \in \mathcal{K} \) and \( \tau_2 \in \mathcal{K} \), so that, by (4.5), (4.6) and (4.18),

\[
(\tau_1, g_1) \in D \quad \text{and} \quad (\tau_2, g_2) \in D.
\]

**Case 1.** \( r = (\sigma, f) \in R_1 \). By (III5) and (4.17), we have

\[ |\varphi_f(\sigma)| = n, \]

so that, by (III7) and interchanging the indices of the elements of \( \sigma \) if necessary, we may assume that

\[
\varphi_{g_1}(\tau_1) \text{ is the unique } \pi - \text{subbalanced subcollection of } \varphi_f(\sigma) \text{ with respect to } (N, n).
\]

Hence, by (4.18), (4.19), (4.20), (R1), (R2) and (R3), we have

\[ D_r = \{ (\tau_1, g_1) \} \quad (r \in R_1), \]

and (4.11) follows.

**Case 2.** \( r = (\sigma, f) \in R_2 \). By (III6) and (4.17), we have

\[ |\varphi_f(\sigma)| = n - 1 \text{ or } n. \]

We discuss two subcases separately.

**Case 2.1.** \( |\varphi_f(\sigma)| = n - 1 \). Then by (III6),

\[
\varphi_f(\sigma) \text{ is a minimal } \pi\text{-subbalanced with respect to } (N, n).
\]

Since

\[ \varphi_f(\sigma) = \{ \varphi_f(v_1)(v_1), \ldots, \varphi_f(v_n)(v_n) \} \]

and \( |\varphi_f(\sigma)| = n - 1 \), we may assume, by interchanging the indices of the elements of \( \sigma \) if necessary, that

\[ \varphi_f(v_{n-1})(v_{n-1}) = \varphi_f(v_n)(v_n), \]

so that

\[
\varphi_{g_1}(\tau_1) = \varphi_{g_2}(\tau_2) = \varphi_f(\sigma).
\]

Hence, by (4.9), (4.18), (4.19), (4.21), (4.22), (R1), (R2) and (R3), we have

\[
D_r = \{ (\tau_1, g_1), (\tau_2, g_2) \}.
\]
Case 2.2. \(|\varphi_f(\sigma)| = n\). Then by (II1) and (II8),

\[(4.24) \quad \varphi_f(\sigma) \text{ contains exactly two minimal } \pi\text{-subbalanced subcollections}
\]

\[\{\varphi_f(v); v \in \tau_1\} \text{ and } \{\varphi_f(v); v \in \tau_2\}\]

with respect to \((N, n)\) for some \(\tau_1 \subset \sigma\) and \(\tau_2 \subset \sigma\), where

\[(4.25) \quad |\tau_1| = |\tau_2| = n - 1 \text{ and } \tau_1 \neq \tau_2.\]

We have

\[n = |\sigma| \geq |\tau_1 \cup \tau_2| = |\tau_1| + |\tau_2| - |\tau_1 \cap \tau_2| = 2(n - 1) - |\tau_1 \cap \tau_2|,\]

so that

\[(4.26) \quad |\tau_1 \cap \tau_2| \geq n - 2.\]

(4.25) and (4.26) implies that

\[(4.27) \quad |\tau_1 \cap \tau_2| = n - 2.\]

By (4.27), We may assume, without loss of generality, that

\[(4.28) \quad \tau_1 = \sigma \setminus \{v_n\} \text{ and } \tau_2 = \sigma \setminus \{v_{n-1}\}.\]

Hence by (4.9), (4.18), (4.19), (4.24), (R1), (R2) and (R3), (4.23) also holds for this subcase, and (4.12) is true.

Case 3. \(r = (\sigma, f) \in R_3\). \(\varphi_f(\sigma)\) contains no \(\pi\)-subbalanced subcollection with respect to \((N, n)\), so that, by (4.9) and (R2)

\[(4.29) \quad D_r = \emptyset \quad (r \in R_3),\]

and (4.13) follows. This proves that (4.11), (4.12) and (4.13) are true, and by the same argument, so are (4.14), (4.15) and (4.16). We next claim that

\[(4.30) \quad |R_d| = m(d \in D_1),\]

\[(4.31) \quad |R_d| = 2m(d \in D_2),\]

\[(4.32) \quad |R_d| = 0(d \in D_3).\]

and in case \(m \geq n\),

\[(4.33) \quad |R_{d^*}| = m - n + 1(d \in D_{1^*}),\]

\[(4.34) \quad |R_{d^*}| = 2(m - n + 1)(d \in D_{2^*}),\]

\[(4.35) \quad |R_{d^*}| = 0(d \in D_{3^*}).\]
To see (4.30) ∼ (4.32), let us fix $d = (\tau, g) \in D$. By (4.5), $|\tau| = n - 1$, we may write

$$\tau = \{v_1, \ldots, v_{n-1}\}.$$  

**Case 1'.** $d = (\tau, g) \in D_1$. Then $\tau \in \partial K$, so that $\tau$ is a face of exactly one $(n - 1)$-simplex $\sigma$ of $K$, say

$$\sigma = \{v_1, \ldots, v_n\}.$$  

Since $M = \{1, \ldots, m\}$, there are exactly $m$ extensions $f_1, \ldots, f_m$ of $g$ to the set $\sigma$ into $M$, where

$$f_k(v_j) = \begin{cases} g(v_j), & \text{if } j = 1, \ldots, n-1 \\ k, & \text{if } j = n \end{cases}$$  

for $k = 1, \ldots, m$. By (4.10), (4.36), (4.37), (4.38), (R1), (R2) and (R3),

$$(4.39) \quad R_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\}$$  

and (4.30) follows.

**Case 2'.** $d = (\tau, g) \in D_2$. Then $\tau \notin \partial K$, so that, by (M1) and (M2), $\tau$ is a face of exactly two distinct $(n - 1)$-simplexes $\sigma$ and $\sigma'$ of $K$, say,

$$\sigma = \tau \cup \{v_n\} \quad \text{and} \quad \sigma' = \tau \cup \{v'_n\}.$$  

For each $k = 1, \ldots, m$, let $f_k : \sigma \to M$ and $f'_k : \sigma' \to M$ be such that

$$f_k|\tau = f'_k|\tau = g \quad \text{and} \quad f_k(v_n) = f'_k(v'_n) = k.$$  

From (4.10), (4.36), (4.40), (4.41), (R1), (R2) and (R3), it follows that

$$R_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \cup \{(\sigma', f'_1), \ldots, (\sigma', f'_m)\},$$  

and (4.31) follows.

**Case 3'.** $d = (\tau, g) \in D_3$. Then $\varphi_g(\tau)$ is not $\pi$-subbalanced with respect to $(N, n)$, so that by (4.10) and (R2),

$$R_d = 0,$$  

and (4.32) follows. This prove (4.30), (4.31) and (4.32). To see (4.33) ∼ (4.35) let us assume that $m \geq n$ and fix $d = (\tau, g) \in D_*$. Since $g$ is one-to-one, so that

$$|g(\tau)| = |\tau| = n - 1.$$
If \( d = (\tau, g) \in D_{1s} \), then there are exactly \( m - n + 1 \) injective extensions of \( g \) to \( \sigma \) into \( M \), namely,

\[
R_{ds} = \{ (\sigma, f_k) \mid k \in M \setminus g(\tau) \} \quad (d \in D_{1s})
\]

where \( \tau, \sigma \), and \( f_k \) are the same as in (4.36), (4.37) and (4.38) respectively. Similarly, if we define \( f_k \) and \( f'_k \) as in (4.41), then we have

\[
R_{ds} = \{ (\sigma, f_k) \mid k \in M \setminus g(\tau) \} \cup \{ (\sigma', f'_k) ; k \in M \setminus g(\tau) \} \quad (d \in D_{2s}).
\]

It is clear that \( R_{ds} = \emptyset \) \( (d \in D_{3s}) \).

This prove (4.33), (4.34) and (4.35). We now claim that

\[
\begin{align*}
R_1 &= K_\pi(\varphi), \\
D_1 &= \partial K_\pi(\varphi), \\
R_{1s} &= K_\pi(\varphi)_{s}, \\
D_{1s} &= \partial K_\pi(\varphi)_{s}.
\end{align*}
\]

For any pair \( (\sigma, f) \), it follows from \( \pi \) is a general position map that the following

\((a) \sim (f)\) are equivalent.

\((a)\) \( (\sigma, f) \in K_\pi(\varphi) \).

\((b)\) \( (\sigma, f) \) is a \( \pi \)-balanced pair with respect to \( N \).

\((c)\) \( \varphi_f(\sigma) \) is a \( \pi \)-balanced collection with respect to \( N \).

\((d)\) \( \sigma \in K, |\sigma| = n, f : \sigma \rightarrow M, \varphi_f(\sigma) \) is a \( \pi \)-balanced with respect to \( N \).

\((e)\) \( (\sigma, f) \in R, \varphi_f(\sigma) \) is a \( \pi \)-balanced with respect to \( N \).

\((f)\) \( (\sigma, f) \in R_1 \)

Thus (4.44) holds. Similarly, for any pair \( (\tau, g) \), the following \((a)' \sim (f)'\) are equivalent.

\((a)'\) \( (\tau, g) \in \partial K_\pi(\varphi) \).

\((b)'\) \( (\tau, g) \) is a \( \pi \)-subbalanced boundary pair with respect to \( (N, n) \).

\((c)'\) \( \varphi_g(\tau) \) is a \( \pi \)-subbalanced collection with respect to \( (N, n), \tau \in \partial K \).

\((d)'\) \( \tau \in K, |\tau| = n - 1, g : \tau \rightarrow M, \varphi_g(\tau) \) is a \( \pi \)-subbalanced with respect to \( (N, n), \tau \in \partial K \).

\((e)'\) \( (\tau, g) \in D, \varphi_g(\tau) \) is a \( \pi \)-subbalanced with respect to \( (N, n), \tau \in \partial K \).

\((f)'\) \( (\tau, g) \in D_1 \)
This shows that (4.45) is true. By the same reason, so are (4.46) and (4.47).

Define \( \lambda: D \times R \rightarrow \{0, 1\} \) by

\[
\lambda(d, r) = \begin{cases} 
1 & \text{if } d \prec r, \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
\sum_{r \in R} \sum_{d \in D} \lambda(d, r) = \sum_{r \in R_1} \sum_{d \in D} \lambda(d, r) + \sum_{r \in R_2} \sum_{d \in D} \lambda(d, r) + \sum_{r \in R_3} \sum_{d \in D} \lambda(d, r)
\]

\[
= \sum_{r \in R_1} |D_r| + \sum_{r \in R_2} |D_r| + \sum_{r \in R_3} |D_r|
\]

\[
= \sum_{r \in R_1} 1 + \sum_{r \in R_2} 2 + \sum_{r \in R_3} 0
\]

\[
= |R_1| + 2|R_2| + 0
\]

\[
= |K_{\pi}(\varphi)| + 2|R_2|
\]

so that

(4.48) \[
\sum_{r \in R} \sum_{d \in D} \lambda(d, r) = |K_{\pi}(\varphi)| + 2|R_2|
\]

and

\[
\sum_{d \in D} \sum_{r \in R} \lambda(d, r) = \sum_{d \in D_1} \sum_{r \in R} \lambda(d, r) + \sum_{d \in D_2} \sum_{r \in R} \lambda(d, r) + \sum_{d \in D_3} \sum_{r \in R} \lambda(d, r)
\]

\[
= \sum_{d \in D_1} |R_d| + \sum_{d \in D_2} |R_d| + \sum_{d \in D_3} |R_d|
\]

\[
= \sum_{d \in D_1} m + \sum_{d \in D_2} 2m + \sum_{d \in D_3} 0
\]

\[
= m|R_1| + 2m|R_2| + 0
\]

\[
= m|\partial K_{\pi}(\varphi)| + 2m|R_2|
\]

so that

(4.49) \[
\sum_{d \in D} \sum_{r \in R} \lambda(d, r) = m|\partial K_{\pi}(\varphi)| + 2m|R_2|
\]

(4.48) and (4.49) imply that

\[
|K_{\pi}(\varphi)| + 2|R_2| = m|\partial K_{\pi}(\varphi)| + 2m|R_2|
\]

this proves (4.1). Similarly, if \( m \geq n \) then we have
(4.50) \[ |\mathcal{K}_\pi(\varphi)_s| + 2|R_{2s}| = (m - n + 1)|\partial\mathcal{K}_\pi(\varphi)_s| + 2(m - n + 1)|D_{2s}| \]

Because of the injectivity, we see that
\[
(4.51) \quad \mathcal{K}_\pi(\varphi)_s = \emptyset \quad \text{if} \quad m < n
\]
\[
(4.52) \quad \partial \mathcal{K}_\pi(\varphi)_s = \emptyset \quad \text{if} \quad m < n - 1
\]
so that both sides of (4.2) are zeros if \( m < n \). Thus, (4.2) follows from (4.50), (4.51) and (4.52). This completes the proof of (4.2).

Suppose further, \((\mathcal{K}, \omega)\) is a coherently oriented \((n - 1)\)-pseudomanifold and \( \omega' = \varepsilon'[a_1, \ldots, a_n] \) (\( \varepsilon' = \pm 1 \)). We claim that
\[
(4.53) \quad \{\mathcal{K}^+_{\pi}(\varphi), \mathcal{K}^-_{\pi}(\varphi)\} \text{ partitions } \mathcal{K}_\pi(\varphi),
\]
\[
(4.54) \quad \{\partial\mathcal{K}^+_{\pi}(\varphi), \partial\mathcal{K}^-_{\pi}(\varphi)\} \text{ partitions } \partial\mathcal{K}_\pi(\varphi),
\]
and
\[
(4.55) \quad \{\mathcal{K}^+_{\pi}(\varphi)_s, \mathcal{K}^-_{\pi}(\varphi)_s\} \text{ partitions } \mathcal{K}_\pi(\varphi)_s,
\]
\[
(4.56) \quad \{\partial\mathcal{K}^+_{\pi}(\varphi)_s, \partial\mathcal{K}^-_{\pi}(\varphi)_s\} \text{ partitions } \partial\mathcal{K}_\pi(\varphi)_s.
\]

Given \((\sigma, f) \in \mathcal{K}_\pi(\varphi)\) with \( \omega(\sigma) = \varepsilon[v_1, \ldots, v_n] \) (\( \varepsilon = \pm 1 \)). Let
\[
(4.57) \quad S_i = \varphi_{f(v_i)}(v_i) \text{ for } i = 1, \ldots, n.
\]

From (a), (c), (2.2), (3.1) and (4.57), it follows, by applying (II9), that (3.2) holds, so that \( \det B \neq 0 \), thus
\[
(\sigma, f) \in \mathcal{K}^+_{\pi}(\varphi) \text{ if and only if } \varepsilon\varepsilon'\det B > 0, \quad \text{and}
\]
\[
(\sigma, f) \in \mathcal{K}^-_{\pi}(\varphi) \text{ if and only if } \varepsilon\varepsilon'\det B < 0.
\]

This proves (4.53). To prove (4.54), let \((\tau, g) \in \partial\mathcal{K}_\pi(\varphi)\) and let \( \sigma \) be the unique \((n - 1)\)-simplex of \( \mathcal{K} \) containing \( \tau \) with \( \omega(\sigma) = \varepsilon[v_1, \ldots, v_n] \) (\( \varepsilon = \pm 1 \)) and \( \tau = \sigma \setminus \{v_n\} \). Then the induced orientation on \( \tau \) from \( \omega(\sigma) \) is
\[
(-1)^{n-1}\varepsilon[v_1, \ldots, v_{n-1}].
\]

Let
\[
(4.58) \quad S_i = \varphi_{g(v_i)}(v_i) \text{ for } i = 1, \ldots, n - 1, \text{ and } S_n = \{n\}.
\]

By \((a)', (c)', (II2)\) with \((P, p) = (N, n), \{S_1, \ldots, S_n\} \) is \( \pi \)-balanced with respect to \( N \), so that, by (2.3) and (II9) with \( B_1 = C, \det B_1 \neq 0 \), thus
\[
(4.59) \quad (\tau, g) \in \partial\mathcal{K}^+_{\pi}(\varphi) \text{ if and only if } (-1)^{n-1}\varepsilon\varepsilon'\det B_1 > 0, \quad \text{and}
\]
\[
(4.60) \quad (\tau, g) \in \partial\mathcal{K}^-_{\pi}(\varphi) \text{ if and only if } (-1)^{n-1}\varepsilon\varepsilon'\det B_1 < 0.
\]
This proves (4.54). By the same reason, (4.55) and (4.56) are also true. Let 
(τ, g) ∼ (σ, f), ω(σ) = ε[v_1, ..., v_n] (ε = ±1), and τ = σ \ {v_n} and (2.2) holds.
we call (τ, g) positive or negative in (σ, f) if

(4.61) \((-1)^{n-1}\varepsilon\varepsilon' det B_1 > 0\)

or

(4.62) \((-1)^{n-1}\varepsilon\varepsilon' det B_1 < 0\)

respectively, where B_1 is the (n − 1) × (n − 1) matrix obtained by deleting the nth row and nth column of B in (2.2).

Put

(4.63) \(D^+_r = \{ d \in D \mid d \prec r, \text{ d is positive in } r \} \) \(r \in R\),

(4.64) \(D^-_r = \{ d \in D \mid d \prec r, \text{ d is negative in } r \} \) \(r \in R\),

(4.65) \(R^+_d = \{ r \in R \mid d \prec r, \text{ d is positive in } r \} \) \(d \in D\),

(4.66) \(R^-_d = \{ r \in R \mid d \prec r, \text{ d is negative in } r \} \) \(d \in D\).

We claim that

(4.67) \(|R^+_d| = m \text{ and } |R^-_d| = 0 \) \(d \in \partial K^+_\pi(\varphi)\),

(4.68) \(|R^+_d| = 0 \text{ and } |R^-_d| = m \) \(d \in \partial K^-_\pi(\varphi)\),

(4.69) \(|R^+_d| = |R^-_d| = m \) \(d \in D_2\),

(4.70) \(|R^+_d| = |R^-_d| = 0 \) \(d \in D_3\),

and if m ≥ n then

(4.71) \(|R^+_{ds}| = m - n + 1 \text{ and } |R^-_{ds}| = 0 \) \(d \in \partial K^+_{\pi}(\varphi)\),

(4.72) \(|R^+_{ds}| = 0 \text{ and } |R^-_{ds}| = m - n + 1 \) \(d \in \partial K^-_{\pi}(\varphi)\),

(4.73) \(|R^+_{ds}| = |R^-_{ds}| = m - n + 1 \) \(d \in D_{2s}\),

(4.74) \(|R^+_{ds}| = |R^-_{ds}| = 0 \) \(d \in D_{3s}\).

If \(d = (\tau, g) \in \partial K^+_{\pi}(\varphi)\) then, from (4.38), (4.39), (4.45), (4.54), (4.59), (4.61),
(4.62), (4.65) and (4.66), it follows that

\(R^+_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \) and \(R^-_d = \emptyset \) \(d \in \partial K^+_{\pi}(\varphi)\).

This proves (4.67). If \(d = (\tau, g) \in \partial K^-_{\pi}(\varphi)\) then from (4.38), (4.39), (4.45), (4.54),
(4.60), (4.61), (4.62), (4.65) and (4.66), it follows that

\(R^+_d = \emptyset \) and \(R^-_d = \{(\sigma, f_1), \ldots, (\sigma, f_m)\} \) \(d \in \partial K^-_{\pi}(\varphi)\).
This proves (4.68). If \( d = (\tau, g) \in D_2 \) then by (C2) and (4.40), \( \omega(\sigma) \) and \( \omega(\sigma') \) induce opposite orientations on \( \tau \), that is, we may assume

\[
\omega(\sigma) = \varepsilon[v_1, \ldots, v_{n-1}, v_n] \quad \text{and} \quad \omega(\sigma') = (-1)\varepsilon[v_1, \ldots, v_{n-1}, v'_n] \quad (\varepsilon = \pm 1),
\]

so, by (4.41), (4.42), (4.61), (4.62), (4.65) and the fact that

\[
(4.76)
\]

This proves (4.69). If \( d = (\tau, g) \in D_3 \) then by, (4.43), (4.65) and (4.66),

\[
R^+_d = R^-_d = \emptyset \quad (d \in D_3).
\]

This proves (4.70). By the same reason, (4.71) \sim (4.74) are also true. We finally claim that

\[
|D^+_r| = \frac{1 + (-1)^{n-1}}{2} \quad \text{and} \quad |D^-_r| = \frac{1 - (-1)^{n-1}}{2} \quad (r \in K^+_\pi(\varphi)), \tag{4.75}
\]

\[
|D^+_r| = \frac{1 - (-1)^{n-1}}{2} \quad \text{and} \quad |D^-_r| = \frac{1 + (-1)^{n-1}}{2} \quad (r \in K^-_\pi(\varphi)), \tag{4.76}
\]

\[
|D^+_r| = |D^-_r| = 1 \quad (r \in R_2), \tag{4.77}
\]

\[
|D^+_r| = |D^-_r| = 0 \quad (r \in R_3), \tag{4.78}
\]

and

\[
|D^+_{rs}| = \frac{1 + (-1)^{n-1}}{2} \quad \text{and} \quad |D^-_{rs}| = \frac{1 - (-1)^{n-1}}{2} \quad (r \in K^+_\pi(\varphi)_s), \tag{4.79}
\]

\[
|D^+_{rs}| = \frac{1 - (-1)^{n-1}}{2} \quad \text{and} \quad |D^-_{rs}| = \frac{1 + (-1)^{n-1}}{2} \quad (r \in K^-_\pi(\varphi)_s), \tag{4.80}
\]

\[
|D^+_{rs}| = |D^-_{rs}| = 1 \quad (r \in R_{2s}), \tag{4.81}
\]

\[
|D^+_{rs}| = |D^-_{rs}| = 0 \quad (r \in R_{3s}). \tag{4.82}
\]

Let \( r = (\sigma, f) \) and \( \omega(\sigma) = \varepsilon[v_1, \ldots, v_n] \) \((\varepsilon = \pm 1)\). If \( r = (\sigma, f) \in K^+_\pi(\varphi) \), then \( \varepsilon \varepsilon' \det B > 0 \) and by (II9) we have \( \varepsilon \varepsilon' \det B_1 > 0 \), thus

\[
(4.73)
\]

\[
(-1)^{n-1}\varepsilon \varepsilon' \det B_1 > 0 \quad \text{if} \ n \text{ is odd},
\]

\[
(-1)^{n-1}\varepsilon \varepsilon' \det B_1 < 0 \quad \text{if} \ n \text{ is even},
\]
so that

\[ D_r^+ = \{(\tau_1, g_1)\} \quad \text{and} \quad D_r^- = \emptyset \quad (r \in K^+_\pi(\varphi)) \text{ if } n \text{ is odd}, \]

\[ D_r^+ = \emptyset \quad \text{and} \quad D_r^- = \{(\tau_1, g_1)\} \quad (r \in K^-_\pi(\varphi)) \text{ if } n \text{ is even}. \]

This proves (4.75). Similarly, we have

\[ D_r^+ = \emptyset \quad \text{and} \quad D_r^- = \{(\tau_1, g_1)\} \quad (r \in K^+_\pi(\varphi)) \text{ if } n \text{ is odd}, \]

\[ D_r^+ = \{(\tau_1, g_1)\} \quad \text{and} \quad D_r^- = \emptyset \quad (r \in K^-_\pi(\varphi)) \text{ if } n \text{ is even}. \]

This proves (4.76). If \( r = (\sigma, f) \in R_2 \) then by (II10), (3.3), (4.18), (4.23) and (4.57), we have

\[ \omega(\sigma) \text{ induces } (-1)^{n-1} \varepsilon [v_1, \ldots, v_{n-1}] \text{ on } \tau_1, \]

\[ \omega(\sigma) \text{ induces } (-1)^{n-2} \varepsilon [v_1, \ldots, v_{n-2}, v_n] \text{ on } \tau_2, \]

\[ (-1)^{n-1} \varepsilon \varepsilon' \det B_1 (-1)^{n-2} \varepsilon \varepsilon' \det B_2 < 0, \]

so that, one of the two pairs \((\tau_1, g_1)\) and \((\tau_2, g_2)\) is positive in \(r\) and the other one is negative in \(r\), thus (4.77) is true. If \( r = (\sigma, f) \in R_3 \) then \( D_r^+ = D_r^- = \emptyset \), this proves (4.78). By the same reason, (4.79) \(\sim\) (4.82) are also true.

Define \( \Lambda : D \times R \to \{-1, 0, 1\} \) by

\[ \Lambda(d, r) = \begin{cases} 
1, & \text{if } d \prec r \text{ and } d \text{ is positive in } r, \\
-1, & \text{if } d \prec r \text{ and } d \text{ is negative in } r, \\
0, & \text{otherwise}.
\end{cases} \]

Then

\[
\sum_{r \in R} \sum_{d \in D} \Lambda(d, r) = \sum_{r \in R} (|D_r^+| - |D_r^-|)
\]

\[
= \left( \sum_{r \in K^+_\pi(\varphi)} + \sum_{r \in K^-_\pi(\varphi)} + \sum_{r \in R_2} + \sum_{r \in R_3} \right) (|D_r^+| - |D_r^-|)
\]

\[
= |K^+_\pi(\varphi)| \left\{ \frac{1 + (-1)^{n-1}}{2} - \frac{1 - (-1)^{n-1}}{2} \right\} + |K^-_\pi(\varphi)| \left\{ \frac{1 - (-1)^{n-1}}{2} - \frac{1 + (-1)^{n-1}}{2} \right\} + |R_2|(1 - 1) + |R_3|(0 - 0)
\]

\[
= (-1)^{n-1} \{|K^+_\pi(\varphi)| - |K^-_\pi(\varphi)|\},
\]

so that

(4.83) \[ \sum_{r \in R} \sum_{d \in D} \Lambda(d, r) = (-1)^{n-1} \{|K^+_\pi(\varphi)| - |K^-_\pi(\varphi)|\} \]
and

\[
\sum_{d \in D} \sum_{r \in R} \Lambda(d, r) = \sum_{d \in D} (|R_d^+| - |R_d^-|)
\]

\[
= \left( \sum_{d \in \partial K^+(\varphi)} + \sum_{d \in \partial K^-(\varphi)} + \sum_{d \in D_2} + \sum_{d \in D_3} \right) (|R_d^+| - |R_d^-|)
\]

\[
= |\partial K^+(\varphi)|(m - 0) + |\partial K^-(\varphi)|(0 - m) + |D_2|(m - m) + |D_3|(0 - 0)
\]

\[
= m\{|\partial K^+(\varphi)| - |\partial K^-(\varphi)|\}
\]

so that

\[
\sum_{d \in D} \sum_{r \in R} \Lambda(d, r) = m\{|\partial K^+(\varphi)| - |\partial K^-(\varphi)|\}
\]

(4.83) and (4.84) imply (4.3). Similarly, if \( m \geq n \) then (4.4) holds. It is clear that if \( m < n \) then both sides of (4.4) are zeros. Thus (4.4) is true. This completes the proof of Theorem 1.

5. Multiple Combinatorial Sperner’s Lemma with Balanced Structure

The following multiple Sperner’s lemma with balanced structure is a consequence of Theorem 1.

**Theorem 2.** Let \( T \) be a triangulation of an \((n - 1)\)-simplex \( A^N = a_1 \ldots a_m \), let \( \varphi : V(T) \to (2^N)^m \) and \( \pi : 2^N \to A^N \), where \( m \) and \( n \) are positive integers, \( N = \{1, \ldots, n\} \) and \( \pi \) is a general position map, such that

(F2) \( \varphi(V(T) \cap A^S) \subset (2^S)^m \) for all \( S \in 2^N \).

Then, for each \( P \in 2^N \), we have

\[
\varphi^P_\mid P\mid = m\mid P\mid,
\]

and, if \( m \geq \mid P\mid \), we have

\[
\varphi^P_\mid P\mid_* = \frac{m!}{(m - \mid P\mid)!}.
\]

We shall apply Theorem 1 and inductive method to prove Theorem 2. The details are as follows.
Let \( P \in 2^N \). Then \( 1 \leq |P| \leq n \). If \( P = \{ p \} \), a singleton, then \( A^P = \overline{a_p} = \{ a_p \} \), so that
\[
H^P = \{ (\{ a_p \}, f_1), \ldots, (\{ a_p \}, f_m) \}
\]
where \( f_i : \{ a_p \} \to M \) \((M = \{ 1, \ldots, m \})\) is the function such that \( f_i(a_p) = i \) for \( i = 1, \ldots, m \). By (F2), we have
\[
\varphi(a_p) = (\{ p \}, \ldots, \{ p \}).
\]
And since \( \pi \) is general position map, \( \pi(\{ p \}) = a_p \) it follows that
\[
\pi(\varphi_i(a_p)) = \pi(\varphi_i(a_p)) = \pi(\{ p \}) = a_p,
\]
which shows that \((\{ a_p \}, f_i)\) is a positively fixed pair under \((\varphi, \pi)\) for \( i = 1, \ldots, m \), thus, we have
\[
\varphi_i^P|_P = |H^P| - 0 = m = m^{|P|} \quad (|P| = 1).
\]
And since each \( f_i \) is one-to-one, we have \( H^P_* = H^P \) then
\[
\varphi_i^P|_* = |H^P_*| - 0 = m = \frac{m!}{(m - |P|)!} \quad (|P| = 1).
\]
Thus the theorem holds for \(|P| = 1\). Suppose now, \( 1 < |P| \leq n \), assume
\[
(5.3) \quad \varphi_i^S = m^{|S|} \quad \text{for all } S \in 2^N \text{ with } 1 \leq |S| < |P|
\]
and
\[
(5.4) \quad \varphi_i^{|S|} = \frac{m!}{(m - |S|)!} \quad \text{for all } S \in 2^N \text{ with } 1 \leq |S| < |P| \text{ if } m \geq |P|.
\]
Fix \( p \in P \). Let \( P = \{ n_1, \ldots, n_{|P|} \} \) where \( n_{|P|} = p \). Then we have
\[
(5.5) \quad A^P = \overline{a_n_1 \ldots a_{n_{|P|}}},
\]
Let
\[
(5.6) \quad K = \{ s \in T \mid s \subset A^P \}.
\]
It is well known, by (T1), (T2), (T3) and (5.6), that \( K \) is a triangulation of \( A^P \) with the vertex scheme
\[
(5.7) \quad \tilde{K} = \{ \sigma \mid \sigma \text{ spans } s \text{ for some } s \in K \}
\]
that is,
\[
s \in K \text{ if and only if } \sigma \in \tilde{K} \quad (\sigma \text{ spans } s).
\]
Let \( \sigma \) be a \((|P| - 1)\)-simplex of \( \tilde{K} \), \( \sigma \) spans \( s \), we may write
\[
\sigma = \{v_1, \ldots, v_{|P|}\} \in \tilde{K}, \quad s = v_1 \ldots v_{|P|} \in K.
\]

Because of the dimension, by (5.5), (5.6) and (5.8), we have
\[
\text{aff}(\sigma) \subset \text{aff}(s) \subset \text{aff}(A^P) \subset \text{aff}(\{a_{n_1}, \ldots, a_{n_{|P|}}\}) \subset \text{aff}(\sigma),
\]
so that
\[
\text{aff}(\sigma) = \text{aff}(A^P)
\]
and
\[
\text{aff}(\{v_1, \ldots, v_{|P|}\}) = \text{aff}(\{a_{n_1}, \ldots, a_{n_{|P|}}\}).
\]

By (5.10), we may write
\[
v_j = \sum_{k=1}^{|P|} \alpha_{jk} a_{nk} \quad (\sum_{k=1}^{|P|} \alpha_{jk} = 1) \quad \text{for} \quad j = 1, \ldots, |P|,
\]
and
\[
a_{nk} = \sum_{j=1}^{|P|} \alpha'_{kj} v_j \quad (\sum_{j=1}^{|P|} \alpha'_{kj} = 1) \quad \text{for} \quad k = 1, \ldots, |P|.
\]

By (5.11), (5.12) and the affine independence of \( \sigma \),
\[
A' = A^{-1}
\]
where \( A \) and \( A' \) are the \(|P| \times |P|\) matrices \((\alpha_{jk})\) and \((\alpha'_{kj})\) in (5.11) and (5.12), respectively. Fix an orientation \( \omega = \varepsilon'[a_{n_1}, \ldots, a_{n_{|P|}}] \) \((\varepsilon' = \pm 1)\). Then, as we mentioned before, \((\tilde{K}, \omega)\) is an coherently oriented \((|P| - 1)\)-pseudomanifold, and by (5.8), (5.11) and compare with (2.1), the orientation \( \omega(\sigma) \) of \( \sigma \) is given by
\[
\omega(\sigma) = \varepsilon'[v_1, \ldots, v_{|P|}] \quad \text{or} \quad \omega(\sigma) = (-1)\varepsilon'[v_1, \ldots, v_{|P|}] \quad \text{if} \quad \det A > 0 \quad \text{or} \quad \det A < 0
\]
respectively, that is,
\[
(5.13) \quad \omega(\sigma) = \varepsilon[v_1, \ldots, v_{|P|}] \quad (\varepsilon = \varepsilon'\det A/|\det A|).
\]

For the given \( \sigma \) in (5.8), let \( f : \sigma \to M \), By (F2) with \( S = P \), we have
\[
(5.14) \quad \varphi_{f(v_i)}(v_i) \subset P \quad \text{for} \quad i = 1, \ldots, |P|,
\]
so that, by (G1) and (5.14),
\begin{equation}
\pi(\varphi_f(v_i)) \in \text{Int}(A^{\varphi_f(v_i)}) \subset A^P \text{ for } i = 1, \ldots, |P|,
\end{equation}
thus, by (5.5) and (5.15), we may write
\begin{equation}
\pi(\varphi_f(v_i)) = \sum_{k=1}^{P} \beta_{ik} \alpha_{nk} \left( \sum_{k=1}^{P} \beta_{ik} = 1 \right) \text{ for } i = 1, \ldots, |P|.
\end{equation}
(5.12) and (5.16) implies that
\begin{equation}
\pi(\varphi_f(v_i)) = \sum_{j=1}^{P} \lambda_{ij} v_j \left( \sum_{j=1}^{P} \lambda_{ij} = 1 \right) \text{ for } i = 1, \ldots, |P|,
\end{equation}
where
\[ \lambda_{ij} = \sum_{k=1}^{P} \beta_{ik} \alpha_{k}^l \text{ for } i = 1, \ldots, |P| \text{ and } j = 1, \ldots, |P|. \]
Thus \( \Lambda = BA^{-1} \) where \( B \) and \( \Lambda \) are the \( |P| \times |P| \) matrices \( (\beta_{ik}) \) and \( (\lambda_{ij}) \) in (5.16) and (5.17) respectively. Compare (2.2) with (5.16) and compare (2.4) with (5.17), we see that the following \( (g) \sim (k) \) are equivalent.

\( (g) \) \( (\sigma, f) \) is a positive (resp. negative) pair.

\( (h) \) \( \varepsilon \varepsilon' detB > 0 \) (resp. \( < 0 \)).

\( (i) \) \( \varepsilon \varepsilon' detA/|detA| \varepsilon \varepsilon' detA > 0 \) (resp. \( < 0 \)).

\( (j) \) \( det\Lambda > 0 \) (resp. \( < 0 \)).

\( (k) \) \( (s, f) \) is a positively (resp. negatively) fixed pair.

We claim that the following \( (l) \) and \( (m) \) are also equivalent.

\( (l) \) \( (\sigma, f) \) is \( \pi \)-balanced pair with respect to \( P \).

\( (m) \) \( (s, f) \) is \( |P| \)-labelled under \((\varphi, \pi)\).

If \( (l) \) holds, then \( \varphi_f(\sigma) \) is \( \pi \)-balanced with respect to \( P \) and, by (5.6), (5.7) and (5.8), we have
\[ s \subset A^P \]
so that, by (L1), \( (m) \) follows. Conversely, if \( (m) \) holds, then by (L1), there exists a \( Q \in 2^N \) such that
\begin{equation}
\text{so that, by (L1), (m) follows. Conversely, if (m) holds, then by (L1), there exists a Q \in 2^N such that}
\end{equation}
\begin{equation}
s \subset A^Q \text{ and } |Q| = |P|, \text{ and}
\end{equation}
\( \varphi_f(\sigma) \) is \( \pi \)-balanced with respect to \( Q \), we have, by (5.18) and the affine independence of \( \sigma \),

\[
 aff(\sigma) \subset aff(s) \subset aff(A^Q) \subset aff(\sigma)
\]

thus

(5.19) \[
 aff(\sigma) = aff(A^Q).
\]

By (5.9), (5.18) and (5.19), we have

(5.20) \[
 A^P = A^Q \quad \text{and} \quad |P| = |Q|,
\]

Since a simplex determines its vertices, we have, by (5.20),

\[ P = Q. \]

So \( \varphi_f(\sigma) \) is \( \pi \)-balanced with respect to \( P \) and \( (l) \) holds. Recall that \( \varphi^P_{|P|} \) is the number of positively fixed \( |P| \)-labelled pairs minus the number of negatively fixed \( |P| \)-labelled pairs under \( (\varphi, \pi) \) in \( H^P \) we have, by the equivalence of \( (g) \) and \( (k) \), by the equivalence of \( (l) \) and \( (m) \), and \( s \subset A^P \), we have

(5.21) \[
 \varphi^P_{|P|} = |\tilde{K}^-_{\pi}(\varphi)| - |\tilde{K}^+_{\pi}(\varphi)|,
\]

and if \( m \geq |P| \), by considering those injective \( f \), we also have

(5.22) \[
 \varphi^P_{(|P| - 1)_*} = |\tilde{K}^-_{\pi}(\varphi)_*| - |\tilde{K}^+_{\pi}(\varphi)_*|,
\]

where the balancedness in (5.21) and (5.22) is the \( \pi \)-balancedness with respect to \( P \).

To apply Theorem 1, we shall prove that

(5.23) \[
 \varphi^P_{|P|-1} = (-1)^{|P|-1} \{ |\partial \tilde{K}^+_{\pi}(\varphi)| - |\partial \tilde{K}^-_{\pi}(\varphi)| \}
\]

and, if \( m \geq |P| \),

(5.24) \[
 \varphi^P_{(|P|-1)_*} = (-1)^{|P|-1} \{ |\partial \tilde{K}^+_{\pi}(\varphi)_*| - |\partial \tilde{K}^-_{\pi}(\varphi)_*| \},
\]

where the subbalancedness in (5.23) and (5.24) is the \( \pi \)-subbalancedness with respect to \( (P, p) \).

Assume that \( \tau \in \tilde{K}, t \in K \) and \( \tau \) spans \( t \). And suppose \( \tau \in \partial \tilde{K} \) if and only if \( t \) is contained in some proper face of \( A^P \) or equivalently, \( \tau \in \partial \tilde{K} \) if and only if \( t \subset A^Q \) for some \( Q \subset P \) with \( |Q| = |P| - 1 \). We claim that the following \( (n) \) and \( (o) \) are equivalent.
(n) \((t, g)\) is \((|P| - 1)\)-labelled under \((\varphi, \pi)\) in \(H^{P \setminus \{p\}}\).
(o) \((\tau, g) \in \partial \widetilde{K}_\pi(\varphi)\).

If \((t, g)\) is \((|P| - 1)\)-labelled under \((\varphi, \pi)\), then, by (L1), there exists a \(Q \in 2^N\) such that \(|Q| = |P| - 1\), \(t \subset A^Q\), and \(\varphi_g(\tau)\) is \(\pi\)-balanced with respect to \(Q\), so that, by (II1) and (II5), we have

\[
(5.25) \quad |\tau| \geq |\varphi_g(\tau)| \geq |P| - 1,
\]
but \(t \subset A^Q\) and \(|Q| = |P| - 1\) imply that

\[
(5.26) \quad |\tau| \leq |P| - 1,
\]
thus, (5.25) and (5.26) imply that

\[
(5.27) \quad |\tau| = |P| - 1.
\]

If (n) holds, then, by the definition of \(H^{P \setminus \{p\}}\),

\[
(5.28) \quad t \subset A^{P \setminus \{p\}}
\]
so that, by (5.27) and (5.28), we must have

\[
(5.29) \quad Q = P \setminus \{p\},
\]
thus \(\varphi_g(\tau)\) is \(\pi\)-balanced with respect to \(P \setminus \{p\}\). Let \(B = \varphi_g(\tau)\). Replacing \(P\) by \(P \setminus \{p\}\) in (B1) and (B2), we have

\[
(5.30) \quad B \subset 2^{P \setminus \{p\}}
\]
and

\[
(5.31) \quad m_{P \setminus \{p\}} \in \text{conv}(\pi(B)).
\]

(5.31) implies that

\[
(5.32) \quad \text{conv}(\pi(B)) \cap (m_p, m_{P \setminus \{p\}}] \neq \emptyset,
\]
thus, by (5.30), (5.32), (SB1) and (SB2), we have \(\varphi_g(\tau)\) is \(\pi\)-subbalanced with respect to \((P, p)\), and since \(\tau \in \partial \widetilde{K}\) and \((\tau, g)\) is \(\pi\)-subbalanced pair with respect to \((P, p)\) that is, (o) holds. This completes the proof of (n) implies (o).

Conversely, if \((\tau, g)\) is \(\pi\)-subbalanced pair with respect to \((P, p)\), there exists a \(v \in A^N\) such that
Let $g \{ (\cdot) \}$ and since

so that, compare (5.38) with (F1),

By comparing (5.39) and (2.4), the following $(p)$ and $(g)$ are equivalent.
\((p)\) \((t, g)\) is positively (resp. negatively) fixed pair under \((\varphi, \pi)\).

\((q)\) \(\det \Lambda_1 > 0\) (resp. < 0).

where \(\Lambda_1\) is the \((|P| - 1) \times (|P| - 1)\) matrix \((\lambda_{ij})\) in (5.39). Let (5.8) and (5.11) hold. Then, by (5.13), \(\omega(\sigma)\) induces the orientation

\[(5.40) \quad (-1)^{|P|-1} \varepsilon[v_1, \ldots, v_{|P|-1}] \quad (\varepsilon = \varepsilon' \det A/|\det A|)\]

on \(\tau\). Note that (5.38) implies that

\[(5.41) \quad \text{aff}(\tau) = \text{aff}(A^{P\setminus \{p\}}) = \text{aff}(\{a_{n_1}, \ldots, a_{n_{|P|-1}}\})\]

by the affine independence of \(\{a_{n_1}, \ldots, a_{n_{|P|}}\}\) and by comparing (5.11) with (5.41), we have

\[(5.42) \quad v_j = \sum_{k=1}^{|P|-1} \alpha_{jk} a_{n_k} \quad (\sum_{k=1}^{|P|-1} \alpha_{jk} = 1) \text{ for } j = 1, \ldots, |P| - 1.\]

this shows that the matrix \(A = (\alpha_{jk})_{|P|\times|P|}\) in (5.11) is of the form

\[(5.43) \quad A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{|P||P|} \end{bmatrix}\]

where \(A_1\) is the \((|P| - 1) \times (|P| - 1)\) matrix \((\alpha_{jk})\) in (5.42). We claim that

\[(5.44) \quad \alpha_{|P||P|} > 0.\]

By the affine independence of \(\sigma = \{v_1, \ldots, v_{|P|}\}\), we have

\[(5.45) \quad v_{|P|} \notin \text{aff}(\tau) \quad (\tau = \{v_1, \ldots, v_{|P|-1}\}).\]

By (5.41) and (5.45), we have

\[(5.46) \quad v_{|P|} \notin \text{aff}(\{a_{n_1}, \ldots, a_{n_{|P|-1}}\}).\]

As \(\sigma = \{v_1, \ldots, v_{|P|}\} \subset A^{P}\) and \(A^{P} = \{a_{n_1}, \ldots, a_{n_{|P|}}\}\), by (5.11) (with \(j = |P|\)) and (5.46), (5.44) follows. Note that (5.39) and (5.42) imply that

\[(5.47) \quad \pi(\varphi_{g(v_i)}(v_i)) = \sum_{k=1}^{|P|-1} \gamma_{ijk} a_{n_k} \quad (\sum_{k=1}^{|P|-1} \gamma_{ijk} = 1) \text{ for } i = 1, \ldots, |P| - 1.\]
where
\[ \gamma_{ik} = \sum_{j=1}^{\vert P \vert - 1} \lambda_{ij} \alpha_{jk} \quad \text{for} \quad i = 1, \ldots, \vert P \vert - 1 \quad \text{and} \quad j = 1, \ldots, \vert P \vert - 1. \]

or equivalently
\[ (5.48) \quad C = \Lambda_1 A_1 \]

where \( C \) is the \((\vert P \vert - 1) \times (\vert P \vert - 1)\) matrix \((\gamma_{ik})\) in (5.47). It follows from (2.3), (5.40), (5.43), (5.44) and (5.48) that the following \((r) \sim (u)\) are equivalent.

\((r)\) \((\tau, g)\) is a positively (resp. negatively) boundary pair.

\((s)\) \((-1)^{\vert P \vert - 1} \varepsilon \varepsilon' \det C > 0\) (resp. \(< 0\)) \((\varepsilon = \varepsilon' \det A/\vert \det A \vert)\).

\((t)\) \((-1)^{\vert P \vert - 1} \varepsilon \varepsilon' \det \Lambda_1 \det A_1 > 0\) (resp. \(< 0\)).

\((u)\) \((-1)^{\vert P \vert - 1} \det \Lambda_1 > 0\) (resp. \(< 0\)).

Thus, by (4.54) and (4.56), by the definition of \( \varphi_{\vert P \vert - 1}^{\{p\}} \) and the equivalences of \((n)\) and \((o)\), \((p)\) and \((q)\), \((r)\) and \((u)\), the formulae (5.23) and (5.24) are true. By (5.21), (5.22), (5.23), (5.24) and Theorem 1, we have
\[ (5.49) \quad \varphi_{\vert P \vert}^P = m \varphi_{\vert P \vert - 1}^{\{p\}}, \]

and, if \( m \geq \vert P \vert \) we have
\[ (5.50) \quad \varphi_{\vert P \vert}^P = (m - \vert P \vert + 1) \varphi_{\vert P \vert - 1}^{\{p\}}. \]

Now, (5.3) and (5.49) imply (5.1), and (5.4) and (5.50) imply (5.2). This completes the inductive proof of Theorem 2.

**Corollary 1.** Let \( T \) be a triangulation of an \((n - 1)\)-simplex \( A^N = a_1 \ldots a_n \), let \( \varphi : V(T) \to (2^N)^m \) and \( \pi : 2^N \to A^N \), where \( m \) and \( n \) are positive integers, \( N = \{1, \ldots, n\} \) and \( \pi \) is a Shapley map, such that
\[ (F2) \quad \varphi(V(T) \cap A^S) \subset (2^S)^m \quad \text{for all} \quad S \in 2^N. \]

Then, there exist at least \( m \vert P \vert \) fixed \( \vert P \vert \)-labelled pairs under \((\varphi, \pi)\) in \( H_P \), and, if \( m \geq \vert P \vert \), there exist at least \( \frac{m!}{(m - \vert P \vert)!} \) fixed \( \vert P \vert \)-labelled pairs under \((\varphi, \pi)\) in \( H_P^* \) for all \( P \in 2^N \).

**Proof.** Since \((S1)\) and \((F2)\) imply \((F1)\), all pairs are fixed under \((\varphi, \pi)\). Now, the assertion follows from (II3), (II4) and Theorem 2.
REFERENCES


Shyh-Nan Lee
Department of Applied Mathematics,
Chung Yuan Christian University,
Chung Li 320, Taiwan
E-mail: nan@math.cycu.edu.tw