THE FORMATION OF SINGULARITIES IN THE HARMONIC MAP HEAT FLOW WITH BOUNDARY CONDITIONS

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Abstract. Let $M$ be a compact manifold with boundary and $N$ be compact manifold without boundary. Let $u(x, t)$ be a smooth solution of the harmonic heat equation from $M$ to $N$ with Dirichlet or Neumann condition. Suppose that $M$ is strictly convex, we will prove a monotonicity formula for $u$. Moreover, if $T$ is the blow-up time for $u$, and $\sup_M |Du|^2(x, t) \leq C/(T - t)$, we prove that a subsequence of the rescaled solutions converges to a homothetically shrinking soliton.

1. INTRODUCTION

Let $M$ and $N$ be compact manifolds and let $u(x, t)$ be a smooth solution of the harmonic heat equation

$$u_t = \Delta_M u + \Gamma_N(u)(Du, Du) \quad \text{in} \quad M \times (0, T).$$

Suppose that $T$ is the blow-up time for $u$, i.e.,

$$\sup_M |Du|(x, t) \to \infty \quad \text{as} \quad t \to T.$$

Let $x_0$ be a singularity point. We define

$$u_{\lambda}(x, t) = u\left(\exp_{x_0} \lambda x, T + \lambda^2 t\right).$$

When $M$ is a compact manifold without boundary and has dimension $n$, in [2], Grayson and Hamilton proved that if the singularity forms rapidly, i.e.,

$$\sup_M |Du|^2(x, t) \leq \frac{C}{T - t},$$

there is a sequence $\lambda_i$ such that on each compact set in $\mathbb{R}^n \times (-\infty, 0)$, the rescaled maps $\{u_{\lambda_i}\}$ converges uniformly to a non-constant map $\bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N$. 
and $\bar{u}$ satisfies the harmonic map heat flow on $\mathbb{R}^n$, and is dilation-invariant, i.e., for any $\lambda > 0$, we have

$$
\bar{u}(x, t) = \bar{u}(\lambda x, \lambda^2 t).
$$

We call a solution of the harmonic heat equation (1.1) satisfying the dilation-invariant condition (1.4) a homothetic soliton.

To prove their results, Grayson and Hamilton made use of a monotonicity formula from [4]: Let $u(x, t) : M \times (0, T) \to N$ be a smooth solution to the harmonic map heat flow, and

$$
\int_M |Du|^2(x, t) \, dx \leq E_0 \quad \text{for} \quad 0 < t < T.
$$

If we define

$$
Z(t) = (T - t) \int_M |Du|^2 k \, dx,
$$

where $k$ is the backward heat kernel on $M$, then, there are constants $B > 0$ and $C > 0$ such that for any $0 < t < T$,

$$
\frac{d}{dt} \left( e^{2C\varphi} Z \right) \leq -2e^{2C\varphi}(T - t) \int_M \left| \Delta u + \frac{Du \cdot Dk}{k} \right|^2 k \, dx + 4CE_0 e^{2C\varphi},
$$

where

$$
\varphi(t) = (T - t) \left( \frac{n}{2} + \log \left( B/(T - t)^{n/2} \right) \right).
$$

This involves a nontrivial estimates on the matrix of second derivatives of the heat kernel on a compact manifold $M$: there are constants $B$ and $C$ depending only on $M$ such that,

$$
D_iD_jk - \frac{D_iD_jk}{k} + \frac{1}{2t}kg_{ij} + Ck \left( 1 + \log \left( Bk/(tm/2) \right) \right) g_{ij} \geq 0.
$$

See [3].

Here, we would like to consider the case where $M$ has non-empty boundary and the solution $u(x, t)$ satisfies the Dirichlet boundary condition

$$
u(x, t) = h(x) \quad \text{on} \quad \partial M \times (0, T)
$$

or the Neumann boundary condition

$$
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M \times (0, T).
$$

Let $x_0$ and $x$ be points in $M$. We denote $r(x_0; x)$ to be the distance between $x_0$ and $x$. We define

$$
E(x_0; t) = (T - t) \int_M |Du|^2(x, t)G(x_0, T; x, t) \, dx,
$$

where $G(x_0, T; x, t)$ is the Green’s function for the Dirichlet problem.
where
\[ G(y, s; x, t) = \left( \frac{1}{4\pi|s-t|} \right)^{n/2} \exp\left( \frac{r^2(y; x)}{4(t-s)} \right). \]

When \( M = \mathbb{R}^n \), the function \( G(y, s; x, t) \) is the backward heat kernel. When \( \partial M \) is strictly convex and \( u(x, t) \) is a smooth solution of the harmonic heat equation and satisfies the Dirichlet boundary condition (1.5), we will prove a monotonicity formula: there is a constant \( A > 0 \), such that
\[
\frac{d}{dt} \left( \exp \left( 2|T - t|^{1/2} \right) \mathcal{E}(t) + A|T - t|^{1/2} \right) \\
\leq -2 \exp \left( 2|T - t|^{1/2} \right) |T - t| \int_M \left( u_t + \frac{Du \cdot D^2 u}{4(t-T)} \right)^2 G \, dx.
\]

Using this formula, we obtain the similar results as in [2]. Let \( u_\lambda \) be the function defined in (1.2). Suppose that (1.3) holds and \((x_0, T)\) is an interior singularity point, then there is a sequence \( \lambda_i \) such that on each compact set in \( \mathbb{R}^n \times (-\infty, 0) \), \( \{u_{\lambda_i}\} \) in converges uniformly to a non-constant map \( \bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N \) and \( \bar{u} \) satisfies the harmonic map heat flow on \( \mathbb{R}^n \), and is dilation-invariant. Let \( \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n > 0 \} \). If \((x_0, T)\) is a boundary singularity point, we show that there is a sequence \( \lambda_i \) such that on each compact set in \( \mathbb{R}^n_+ \times (-\infty, 0) \), \( \{u_{\lambda_i}\} \) in converges uniformly to a non-constant map \( \bar{u} : \mathbb{R}^n_+ \times (-\infty, 0) \to N \). Also, the limit function \( \bar{u} \) satisfies the harmonic map heat flow on \( \mathbb{R}^n_+ \times (-\infty, 0) \), and is dilation-invariant, and is a constant on the hyperplane \( \{(x, t) \in \mathbb{R}^n \times (-\infty, 0) : x_n = 0\} \).

It is interesting to know whether boundary singularities exist. This is equivalent to ask whether there is non-constant solution to the harmonic map heat flow on \( \mathbb{R}^n_+ \times (-\infty, 0) \), and is dilation-invariant and is a constant on the hyperplane \( \{(x, t) \in \mathbb{R}^n_+ \times (-\infty, 0) : x_n = 0\} \). In fact, there are harmonic maps from \( B^3(1) = \{ x \in \mathbb{R}^3 : |x| < 1 \} \) to \( S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \} \) which is smooth in \( B^3 \) and have singularities on the boundary, [6].

Let \( u : M \times [0, T) \to N \) be a regular solution of (1.1) with Neumann boundary condition (1.6). Suppose that \( M \) is a compact manifold with convex boundary. We prove that similar results are true. Let \( \mathcal{E}(x_0; t) \) be the energy function defined in the above, we show that there is a constant \( B > 0 \) such that
\[
\frac{d}{dt} \left( \exp \left( 2|T - t|^{1/2} \right) \mathcal{E}(t) + B|T - t|^{1/2} \right) \\
\leq -2 \exp \left( 2|T - t|^{1/2} \right) |T - t| \int_M \left( u_t + \frac{Du \cdot D^2 u}{4(t-T)} \right)^2 G \, dx.
\]

Using this monotonicity formula, it is not difficult to see that the small-energy-regularity theory also works and the rescaled solution converges to a homothetically shrinking solititon solution.
In a forthcoming paper, we will use similar method to treat the equation

\[ u_t = \Delta u + u^p \]

defined on a compact manifold with convex boundary.

2. Monotonicity Formula

Let \( M \) be a compact manifold with \( C^{2,\alpha} \) boundary and \( N \) be a compact manifold. Let \( u(x, t) \) be a smooth solution of the harmonic heat equation

\[
(2.1) \quad u_t = \Delta M u + \Gamma_N(u)(D u, D u) \quad \text{in} \quad M \times (0, T).
\]

The term \( \Gamma_N(u)(D u, D u) \) is perpendicular to the tangent plane at \( u(x) \) and for some constant \( C > 0 \), depending only on \( N \),

\[
|\Gamma_N(u)(D u, D u)| \leq C |D u|^2.
\]

We assume that \( u(x, t) \) satisfies the Dirichlet boundary condition

\[
(2.2) \quad u(x, t) = h(x) \quad \text{on} \quad \partial M \times (0, T)
\]

where \( h \) is a function in \( C^{2,\alpha}(\overline{M}, N) \). Let \( x \) and \( x_0 \) be in \( \overline{M} \). We denote \( r(x_0; x) \) to be the distance between \( x_0 \) and \( x \) on \( M \). We say \( \partial M \) is strictly convex, if there is a constant \( \gamma > 0 \) so that for any \( x_0 \in \overline{M} \),

\[
(2.3) \quad Dr^2 \cdot \nu \geq \gamma r^2 > 0 \quad \text{on} \quad \partial M,
\]

where \( \nu \) is the unit outward normal on \( \partial M \).

Suppose that \( \Omega \) is a strictly convex domain in \( \mathbb{R}^n \) with smooth boundary. There exists \( R > 0 \) such that for any \( x \in \partial \Omega \), there is \( y \in \mathbb{R}^n \), \( \Omega \) is contained in \( B(y, R) = \{ x : |x - y| < R \} \) and \( \partial B(y, R) \cap \partial \Omega = \{ x \} \). In that case, if \( v(x) \) is the unit outward normal at \( x \), then we have \( v(x) = (x - y)/|x - y| \). Also, for any \( x_0 \in \overline{\Omega} \), we have \( r(x, x_0) = |x - x_0| \) and \( Dr^2(x, x_0) = 2(x - x_0) \). Thus,

\[
Dr^2(x, x_0) \cdot \nu(x) = 2 \frac{(x - x_0) \cdot (x - y)}{|x - y|} = \frac{2|x - y|^2 - 2(x_0 - y) \cdot (x - y)}{|x - y|}.
\]

Since \( |x - y| = R \) and \( |x_0 - y| \leq R \), we have

\[
Dr^2(x, x_0) \cdot \nu(x) \geq \frac{|x - y|^2 - 2(x_0 - y) \cdot (x - y) + |x_0 - y|^2}{|x - y|} = \frac{r^2(x, x_0)}{R}.
\]

Hence, (2.3) is true with \( \gamma = 1/R \).

For any \( x_0 \in M \), we also define the function

\[
G(x_0, T; x, t) = \left( \frac{1}{4\pi|T - t|} \right)^{n/2} \exp\left( \frac{r^2(x_0; x)}{4(t - T)} \right).
\]
Suppose that
\[ \max_{x \in M} |Du|(x, t) \to \infty \quad \text{as} \quad t \to T. \]

For any \( x_0 \in \bar{M} \), let
\[ \mathcal{E}(x_0; t) = (T - t) \int_M |Du|^2(x, t)G(x_0, T; x, t)\, dx. \]

**Theorem 2.1.** Suppose that \( \partial M \) is strictly convex. Let \( u(x, t) \) be a smooth solution of the harmonic heat equation with Dirichlet boundary condition, and
\[ (2.4) \quad \int_M |Du|^2(x, t)\, dx \leq E_0 \quad \text{for} \quad t \in (0, T). \]

Then, there is a constant \( A > 0 \), depending only on \( M, N, h, T \) and \( E_0 \), so that, for all \( t \in (0, T) \),
\[ \frac{d}{dt} \left( \exp \left( 2|T - t|^{1/2} \right) \mathcal{E}(x_0; t) + A|T - t|^{1/2} \right) \leq -2 \exp \left( 2|T - t|^{1/2} \right) |T - t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4(t - T)} \right)^2 G(x_0, T; x, t)\, dx. \]

We will need the following propositions. The first one concerns the hessian of the distance function, the second one concerns an integral on the boundary.

**Proposition 2.2.** Let \( x_0 \in \bar{M} \) and \( r(x) = \text{dist}(x, x_0) \). There is a constant \( C \) depending on \( M \) so that
\[ |\Delta r^2 - 2n| \leq Cr^2 \]
and
\[ |D^2(r^2)(X, X) - 2|X|^2| \leq Cr^2|X|^2, \]
where \( D^2(f) \) denotes the hessian of a function \( f \) and \( X \) is any tangent vector on \( T_xM \).

**Proposition 2.3.** There is a constant \( C > 0 \), depending on the geometries of \( \partial M \) and \( M \) only, so that, for any \( x_0 \in \bar{M} \),
\[ \int_{\partial M} G(x_0, T; x, t)\, d\sigma \leq \frac{C}{|t|^{1/2}}. \]

**Proof.** Since \( \partial M \) is \( C^{2,\alpha} \) and compact, there is \( R > 0 \) such that for any \( a \in M \), and \( \text{dist}(a, \partial M) < R \), there is \( \tilde{a} \in \partial M \) such that \( \text{dist}(a, \partial M) = \text{dist}(a, \tilde{a}) \). Moreover, we may choose \( R \) small enough, such that for each \( \tilde{a} \in \partial M \), the set
\[ B(\tilde{a}, R) = \{ x \in \bar{M} : \text{dist}(x, \tilde{a}) < R \} \]
can be represented by a chart \((\phi_1, \ldots, \phi_n)\) so that \( B(\tilde{a}, R) \cap M \) is identified with a region \( \Omega \).
\[ \Omega \subset \{ \phi \in \mathbb{R}^n : |\phi| \leq 2R, \ \phi_n > \varphi(\phi_1, ..., \phi_{n-1}) \}, \]

for some \( C^{2,\alpha} \) function \( \varphi, \ \varphi(0) = 0 \). The boundary region \( \partial M \cap B(\bar{a}, R) \) is identified with the graph \( \phi_n = \varphi(\phi_1, ..., \phi_n) \) and the point \( \bar{a} \) is identified with the point \( 0 \in \mathbb{R}^n \). Since \( \partial M \) is a compact set, if \( R \) is chosen small enough, there is a constant \( \delta > 0 \), depending only on \( M \), such that if \( x, \bar{x} \in B(\bar{a}, R) \cap M \), and \( \phi, \bar{\phi} \) be corresponding points in \( \Omega \), we have

\[ \delta \text{dist}_M(x, \bar{x}) \leq \text{dist}_{\mathbb{R}^n}(\phi, \bar{\phi}) \leq \frac{1}{\delta} \text{dist}_M(x, \bar{x}). \]

Furthermore, if we choose \( R \) and \( \delta \) small enough, for \( x, \bar{x} \in \partial M \cap B(\bar{a}, R) \), we also have

\[ \delta \text{dist}_{\partial M}(x, \bar{x}) \leq \text{dist}_{\mathbb{R}^n}(\phi, \bar{\phi}) \leq \frac{1}{\delta} \text{dist}_{\partial M}(x, \bar{x}). \]

Now, let \( x_0 \in \bar{M} \) and \( \text{dist}(x_0, \partial M) = d < R/2 \). We can find \( \bar{x}_0 \in \partial M \) and a chart \((\phi_1, ..., \phi_n) \) described in the above. After a rotation, we may assume that the point \( \bar{x}_0 \) is identified with the origin in the chart and the point \( x_0 \) is identified with the point \((0, ..., 0, d) \). For any \( x \in \partial M \cap B(\bar{x}_0, R) \), which is identified with a point \( \phi \in \partial \Omega \), we have

\[
\frac{1}{\delta} \text{dist}_M^2(x, x_0) \geq \phi_1^2 + ... + \phi_{n-1}^2 + (\phi_n - d)^2 \geq \phi_1^2 + ... + \phi_{n-1}^2 \geq \delta \text{dist}_{\partial M}^2(x, \bar{x}_0) \geq \delta \text{dist}_M^2(x, \bar{x}_0).
\]

We let \( \tilde{r}(x) = \text{dist}_{\partial M}^2(x, \bar{x}_0) \) for \( x \in \partial M \). Then,

\[
G(x, t) \leq \frac{1}{|t|^{n/2}} \exp \left( \frac{\delta^2 \tilde{r}^2(x)}{4t} \right) \quad \text{when } x \in \partial M \cap B(\bar{x}_0, R), \quad t < 0,
\]

and

\[
G(x, t) \leq \frac{1}{|t|^{n/2}} \exp \left( \frac{R^2}{4t} \right) \quad \text{when } x \in \partial M - B(\bar{x}_0, R), \quad t < 0.
\]

Thus, when \( \text{dist}(x_0, \partial M) \leq R/2 \), we have

\[
\int_{\partial M} G \ d\sigma = \int_{\partial M \cap B(\bar{x}_0, R)} G \ d\sigma + \int_{\partial M - B(\bar{x}_0, R)} G \ d\sigma
\]

\[
\leq \frac{C_2}{|t|^{1/2}} + \frac{1}{|t|^{n/2}} \exp \left( \frac{R}{4t} \right) \text{vol}(\partial M)
\]

\[
\leq \frac{C_3}{|t|^{1/2}}.
\]

If \( \text{dist}(x_0, \partial M) > R/2 \), then

\[
\int_{\partial M} G \ d\sigma = \frac{1}{|t|^{n/2}} \int_{\partial M} \exp \left( \frac{r^2}{4t} \right) d\sigma \leq \frac{1}{|t|^{n/2}} \exp \left( \frac{R^2}{16t} \right) \text{vol}(\partial M).
\]
From (2.6) and (2.7), there is a constant $C_4 > 0$ so that

$$\int_{\partial M} G \, d\sigma \leq \frac{C_4}{|t|^{1/2}}. \tag{2.8}$$

We note that the constant $C_4$ depends on the geometries of $\partial M$ and $M$ only. ■

**Proof of Theorem 2.1.** After a translation in time, we may assume the $u(x,t)$ is defined on $(-T,0)$. Let $x_0 \in M$. We will write $r(x) = r(x_0; x) = \text{dist}(x_0, x)$, and

$$E(t) = E(x_0; t) = |t| \int_M |Du|^2(x,t)G(x,t) \, dx,$$

where

$$G(x,t) = \left( \frac{1}{4\pi|t|} \right)^{n/2} \exp \left( \frac{r^2(x)}{4t} \right),$$

for $x \in M$ and $t \in (-T,0)$. By straightforward computations, we have

$$E'(t)$$

$$= -\int_M |Du|^2(x,t)G(x,t) \, dx + |t| \int_M \left( 2Du \cdot Du_t G + |Du|^2 G_t \right) \, dx$$

$$= -\int_M |Du|^2(x,t)G(x,t) \, dx + 2|t| \int_M \left( Du \cdot Du_t + \frac{Du \cdot D^2 u \cdot Dr^2}{4t} \right) G \, dx$$

$$+ |t| \int_M |Du|^2(G_t + \Delta G) \, dx + 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G \, d\sigma$$

$$= -\int_M |Du|^2(x,t)G(x,t) \, dx + 2|t| \int_M Du \cdot D \left( u_t + \frac{Du \cdot Dr^2}{4t} \right) G \, dx$$

$$- 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G \, d\sigma$$

$$= -2|t| \int_M \left( \Delta u + \frac{Du \cdot Dr^2}{4t} \right) \left( u_t + \frac{Du \cdot Dr^2}{4t} \right) G \, dx$$

$$- \int_M |Du|^2(x,t)G(x,t) \, dx - 2|t| \int_M \frac{Du \cdot D^2 r^2 \cdot Du}{4t} G \, dx$$

$$+ |t| \int_M |Du|^2(G_t + \Delta G) \, dx + 2|t| \int_{\partial M} |Du|^2 \frac{Dr^2 \cdot \nu}{4t} G \, d\sigma$$

$$+ 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu} \left( u_t + \frac{Du \cdot Dr^2}{4t} \right) G \, d\sigma.$$

By equation (2.1), since the term $\Gamma_N(u)(Du, Du)$ is orthogonal to $T_{u(x)}N$, we have
Also, by (2.3), we have

$$
(2.10)
\partial u
$$

Then,

$$
\partial u
\in \gamma \leq - \left( 2 - \frac{1}{2} | \int_M \frac{Du}{4t} G dx \right) dx
$$

Since $u_t = 0$ on $\partial M$, from (2.9), we have

$$
(2.9)
\partial u
$$

On $\partial M$, we may write

$$
Du = \frac{\partial u}{\partial \nu} + D_T u \quad \text{and} \quad Dr^2 = Dr^2 \cdot \nu + D_T r^2.
$$

Then,

$$
\frac{\partial u}{\partial \nu} (Du \cdot Dr^2) = \frac{\partial u}{\partial \nu} \left( \frac{Du}{4t} (Dr^2 \cdot \nu) + D_T u \cdot D_T r^2 \right).
$$

When $t \in (-T, 0)$, this gives

$$
2|t| \int_{\partial M} |Du| \frac{Du}{4t} (Dr^2 \cdot \nu) G d\sigma + 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu} \frac{Du}{4t} (Dr^2 \cdot \nu) G d\sigma
$$

$$
= - \frac{1}{2} \int_{\partial M} (Du^2 (Dr^2 \cdot \nu)) G d\sigma - \frac{1}{2} \int_{\partial M} \frac{\partial u}{\partial \nu} (D_T u \cdot D_T r^2) G d\sigma
$$

$$
- \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 (Dr^2 \cdot \nu) G d\sigma
$$

Also, by (2.3), we have

$$
2|t| \int_{\partial M} |Du| \frac{Du}{4t} (Dr^2 \cdot \nu) G d\sigma + 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu} \frac{Du}{4t} (Dr^2 \cdot \nu) G d\sigma
$$

$$
\leq - \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 (Dr^2 \cdot \nu) G d\sigma + \frac{1}{2} \int_{\partial M} \left| \frac{\partial u}{\partial \nu} \right| (Du^2 (Dr^2 \cdot \nu)) G d\sigma
$$

$$
\leq - \gamma \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 G d\sigma + \int_{\partial M} \left| \frac{\partial u}{\partial \nu} \right| (Du^2 (Dr^2 \cdot \nu)) G d\sigma
$$

$$
\leq - \gamma \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 G d\sigma + \int_{\partial M} \left| \frac{\partial u}{\partial \nu} \right| (Du^2 (Dr^2 \cdot \nu) G d\sigma
$$
\[ \leq -\gamma \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 r^2 G \, d\sigma + \gamma \int_{\partial M} \left( \frac{\partial u}{\partial \nu} \right)^2 r^2 G \, d\sigma + \frac{1}{4\gamma} \int_{\partial M} |D_T u|^2 |D_T r^2 G \, d\sigma \]
\[ \leq \frac{1}{4\gamma} \int_{\partial M} |D_T u|^2 |D_T r^2 G \, d\sigma \]

Thus, one can see that there is a constant \( C_1 \), depending only on \( h \) and \( \gamma \) and the geometries of \( \partial M \) and \( M \), so that

\[
2|t| \int_{\partial M} |D u|^2 \frac{D^2 r \cdot D}{4t} G \, d\sigma + 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu} \frac{D u \cdot D^2 r}{4t} G \, d\sigma \leq \frac{C_5}{|t|^{1/2}}.
\]

By Proposition 2.3, we obtain

\[
2|t| \int_{\partial M} |D u|^2 \frac{D^2 r \cdot D}{4t} G \, d\sigma + 2|t| \int_{\partial M} \frac{\partial u}{\partial \nu} \frac{D u \cdot D^2 r}{4t} G \, d\sigma \leq \frac{C_5}{|t|^{1/2}}.
\]

Then, equation (2.10) becomes

\[
E'(t) \leq -2|t| \int_{\partial M} \left( u + \frac{D u \cdot D^2 r}{4t} \right)^2 G \, dx + |t| \int_{\partial M} |D u|^2 (G_1 + \Delta G) \, dx
\]
\[- \int_{\partial M} |D u|^2 (x, t) G(x, t) \, dx - 2|t| \int_{\partial M} \frac{D u \cdot D^2 r \cdot D u}{4t} G \, dx + \frac{C_5}{|t|^{1/2}}.
\]

On the other hand, it is easy to compute that

\[
G_1 + \Delta G = \left( -\frac{n}{2t} + \frac{\Delta r^2}{4t} \right) G.
\]

By Proposition 2.2, we have

\[
|G_1 + \Delta G| \leq C_6 \frac{r^2}{|t|} G
\]

and

\[
\left| \frac{|D u|^2}{|t|} + \frac{D_i u D_j r^2 D_j u}{2t} \right| \leq C_7 \frac{r^2}{|t|} |D u|^2.
\]

Let \( t \) be fixed and \( \Gamma = \{ x \in M : r^2(x) < |t|^{1/2} \} \). Then,

\[
\int_M |D u|^2 (x, t) \frac{r^2}{|t|} G(x, t) \, dx
\]
\[
= \int_\Gamma |D u|^2 (x, t) \frac{r^2}{|t|} G(x, t) \, dx + \int_{M - \Gamma} |D u|^2 (x, t) \frac{r^2}{|t|} G(x, t) \, dx
\]
\[
\leq \frac{1}{|t|^{1/2}} \int_M |D u|^2 (x, t) G(x, t) \, dx + \int_M |D u|^2 \frac{r^2}{|t|^{1/2}} \frac{1}{2|t|^{1/2}} \exp \left( -\frac{1}{4|t|^{1/2}} \right) \, dx
\]
\[
\leq \frac{1}{|t|^{1/2}} \int_M |Du|^2(x,t)G(x,t) \, dx + C_8 \exp \left( -\frac{1}{8|t|^{1/2}} \right) \int_M |Du|^2 \, dx.
\]

Thus, by (2.4), we have
\[
\int_M |Du|^2(x,t) \frac{r^2}{|t|} G(x,t) \, dx \leq \frac{1}{|t|^{1/2}} \int_M |Du|^2(x,t)G(x,t) \, dx + C_9 \exp \left( -\frac{1}{8|t|^{1/2}} \right).
\]

(2.15)

Combining (2.12), (2.13), (2.14) and (2.15), we have
\[
\mathcal{E}'(t) \leq -\frac{2|t|}{4t} \left( u_t + \frac{Du \cdot Dr^2}{4t} \right)^2 G \, dx + \frac{1}{|t|^{1/2}} \mathcal{E}(t) + \frac{C_{10}}{|t|^{1/2}}.
\]

The constant \( C_{10} \) depends only on \( M, N, h \) and \( E_0 \) only. It follows that, for \( t \in (-T,0) \),
\[
\frac{d}{dt} \left( \exp \left( \frac{2|t|^{1/2}}{} \right) \mathcal{E}(t) \right) \leq -2 \exp \left( \frac{2|t|^{1/2}}{} \right) |t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4t} \right)^2 G \, dx + \frac{C_{10}}{|t|^{1/2}}.
\]

By choosing a constant \( A > 0 \) large enough, one sees that, for \( t \in (-T,0) \),
\[
\frac{d}{dt} \left( \exp \left( \frac{2|t|^{1/2}}{} \right) \mathcal{E}(t) + A|t|^{1/2} \right) \leq -2 \exp \left( \frac{2|t|^{1/2}}{} \right) |t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4t} \right)^2 G \, dx.
\]

This completes the proof.

3. Partial Regularity Results

Let \( u : M \times [-4R_0^2,0] \to N \) be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Let \( x_0 \in \bar{M} \) be fixed. Let
\[
r(x) = \text{dist}_M(x,x_0),
\]
\[
P(R)(x_0) = \{(x,t) : x \in M, \ r(x) < R, \ t \in (-R^2,0)\},
\]
\[
T(R)(x_0) = \{(x,t) : x \in M, \ r(x) < R, \ t \in (-4R^2,-R^2)\}.
\]
Lemma 3.1. Let $u : M \times [-1, 0] \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Suppose that for some $A > 0$,

$$|Du|^2(x, t) \leq A \quad \text{on} \quad P(2R).$$

Then, if $x_0 \in \bar{M}$ and $R > 0$ and $R$ is less than the injectivity radius on $M$, then

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/8))} \leq C(A + \|h\|_{C^{2+\alpha}(\partial M)}).$$

Proof. We first assume that $\text{dist}(x_0, \partial M) > R/4$. We note that in equation (2.1), we have

$$|\Gamma_N(u)(Du, Du)| \leq C|Du|^2.$$

By interior regularity theory, ([5], Chap. IV, Theorem 9.1), for any $q > 1$,

$$\|u\|_{W^{2,1}(P(R/2))} \leq CA,$$

where for any $Q \subset \mathbb{R}^n \times \mathbb{R}$, and $q > 1$,

$$\|u\|_{W^{2,1,q}(Q)} = \left(\int \int_Q (|u|^q + |D^2u|^q + |Du|^q + |u|^q) \, dx \, dt\right)^{1/q}.$$

We choose $q > (n+2)/(1 - \alpha)$. Then, by the Sobolev inequality, Lemma 3.3, Chapter II, [5], $Du \in C^{\alpha,\alpha/2}(P(R/4))$ and

$$\|Du\|_{C^{\alpha,\alpha/2}(P(R/4))} \leq C\|u\|_{W^{2,1}(P(R/2))} \leq CA.$$

It follows from the parabolic Schauder’s estimates that

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/8))} \leq CA.$$

Suppose that $x_0 \in \partial M$. For any $q > 1$, by the boundary regularity theory, we have

$$\|u\|_{W^{2,1,q}(P(2R))} \leq C(A + \|h\|_{C^2(\partial M)}).$$

We choose $q > (n+2)/(1 - \alpha)$. Then, by the Sobolev inequality, Lemma 3.3, Chapter II, [5], $Du \in C^{\alpha,\alpha/2}(P(R))$ and

$$\|Du\|_{C^{\alpha,\alpha/2}(P(R))} \leq C\|u\|_{W^{2,1,q}(P(2R))} \leq C(A + \|h\|_{C^2(\partial M)}).$$

It follows from the parabolic Schauder’s estimates that

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(P(R/2))} \leq C(A + \|h\|_{C^{2+\alpha}(\partial M)}).$$

If $x_0 \in M$ and $\text{dist}(x_0, \partial M) \leq R/4$, we can choose $x_1 \in \partial M$ such that $P(R/8)(x_0) \subset P(R/2))(x_1)$. Then we obtain Lemma 3.1. ■
Corollary 3.2. Let \( u : M \times [0, T) \to N \) be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Suppose that for some \( C_1 > 0 \), we have

\[
\sup_{x \in M} |Du|^2(x, t) \leq \frac{C_1}{T - t}.
\]

Then there is a constant \( C_2 > 0 \) so that

\[
\sup_{x \in M} \left( |D^2u(x, t) + |u_t(x, t)| \right) \leq \frac{C_2}{T - t}.
\]

As in the previous section, for any \( x_0 \in \overline{M} \), we let \( r(x) = \text{dist}(x, x_0) \) and

\[
G(x, t) = \left( \frac{1}{4\pi|t|} \right)^{n/2} \exp \left( \frac{r^2(x)}{4t} \right).
\]

In [1], Y. Chen proved that

Lemma 3.3. Suppose that \( M \) is a compact manifold with non-empty boundary. There is a constant \( \epsilon_1 > 0 \) depending only on \( M, N \) and \( h \) only, such that for any regular solution \( u : M \times [-4R_0^2, 0] \to N \) of (2.1) with Dirichlet boundary condition (2.2) and

\[
\int_M |Du|^2(x, t) \, dx \leq E_0 < \infty, \quad \text{for } t \in [-4R_0^2, 0),
\]

the following is true: If for some \( R \in (0, R_0) \) there holds

\[
\int_{T(R)} |Du|^2 G \, dx \, dt < \epsilon_1,
\]

then there are constants \( \delta > 0 \), depending on \( M, N, h, E_0, \) and \( R \) only, and \( C > 0 \) depending on \( M, N \) and \( h \) only, so that

\[
\sup_{P(\delta R)} |Du|^2 \leq C(\delta R)^{-2}.
\]

From Chen’s result, we have

Theorem 3.4. Suppose that \( M \) is a compact manifold with strictly convex boundary. There are constants \( \epsilon_2 > 0 \) and \( \beta > 0 \), depending only on \( M, N \) and \( h \) only, such that for any regular solution \( u : M \times [-T, 0) \to N \) of (2.1) with Dirichlet boundary condition (2.2) and

\[
\int_M |Du|^2(x, t) \, dx \leq E_0 < \infty, \quad \text{for } t \in [-T, 0),
\]

the following is true: If

\[
|t_0| \int_M |Du|^2(x, t_0)G(x, t_0) \, dx < \epsilon_2
\]
for some $t_0 \in (-\beta, 0)$, then there are constants $\delta > 0$, depending on $M$, $N$, $E_0$, and $\beta$ only, and $C > 0$ depending on $M$, $N$ only, so that

$$\sup_{P(\delta \sqrt{|t_0|})} |Du|^2 \leq \frac{C}{\delta^2 |t_0|}.$$  

**Proof.** Let $t_0 = -4R^2$. If $x_0$ lies in the interior of $M$ and $\text{dist}(x_0, \partial M) > R$, using the monotonicity formula (2.5), we may follow the arguments in [2] to prove the Theorem.

Suppose that $\text{dist}(x_0, \partial M) \leq R$. By the monotonicity formula (2.5), if (3.2) holds, there is $C_1 > 0$ so that

$$\int_{T(R)} |Du|^2 G \, dx \, dt \leq \int_{-4R^2}^{R} \int_{r(x) < R} |Du|^2 G \, dx \, dt \leq \frac{1}{4R^2} \int_{-4R^2}^{-R^2} |t| \int_M |Du|^2 G \, dx \, dt \leq C_1 \epsilon_2.$$

If $\epsilon_2$ is chosen small enough, by Lemma 3.3, Theorem 3.4 follows. 

Let $S$ be a subset in $M$. We denote the $k$-dimensional Hausdorff measure of $S$ by $\mathcal{H}_k(S)$. As in [2], using Theorem 3.4, we can prove that

**Theorem 3.5.** Suppose that $M$ is a compact manifold with strictly convex boundary. Let $u : M \times [0, T) \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2) and

$$\int_M |Du|^2(x, t) \, dx \leq E_0 < \infty, \quad \text{for} \quad t \in [0, T).$$

Let $n$ be the dimension of $M$. Then, there exists a closed set $S$ with finite $n-2$ dimensional measure such that $u(x, t)$ converges smoothly to a limit $u(x, T)$ as $t \to T$ on compact sets in $M - S$. Moreover, there exists a constant $C > 0$ depending only on $M$, $N$, $h$ and $E_0$ such that if $U$ is any relatively open set containing $S$, then

$$\mathcal{H}_{n-2}(S) \leq C \liminf_{t \to T} \int_U |Du|^2(x, t) \, dx.$$

4. **Convergence to the Homothetically Shrinking Solution**

Let $M$ be a compact manifold with non-empty $C^{2,\alpha}$, strictly convex boundary. Let $u : M \times [0, T) \to N$ be a regular solution of (2.1) with Dirichlet boundary condition (2.2). We assume that there is a constant $C_1 > 0$ so that

$$\sup_{x \in M} |Du|^2(x, t) \leq \frac{C_1}{T-t}.$$
We denote
\[ B(R) = \{ x \in M : \text{dist}(x, x_0) < R \} \]
and
\[ P(R) = \{ (x, t) \in M \times (0, T) : \text{dist}(x, x_0) < R, \ t \in (T - R^2, T) \}. \]

Let \((x_0, T)\) be an interior singularity, i.e., \(x_0 \in M\) and there are sequences \(x_n \in M\) and \(t_n \in (0, T)\), such that \(x_n \to x_0\) and \(T_n \to T\) as \(n \to \infty\), and
\[ \lim_{n \to \infty} |D u|(x_n, t_n) = \infty. \]

We let\[ u_{\lambda}(x, t) = u \left( \exp_{x_0} \lambda x, T + \lambda^2 t \right). \]

Using almost the same arguments as in [2], we can show that there is a sequence \(\lambda_i\) such that on each compact set in \(\mathbb{R}^n \times (-\infty, 0)\), \(\{ u_{\lambda_i} \}\) in \(C^\infty\) converges to a non-constant map\[ \bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N \]
and \(\bar{u}\) satisfies the harmonic map heat flow, and is dilation-invariant, i.e., for any \(\lambda > 0\), we have \(\bar{u}(x, t) = \bar{u}(\lambda x, \lambda^2 t)\).

Now we examine the boundary singularities in greater detail by blowing them up. Let \(u : M \times [0, T) \to N\) be a regular solution of (2.1) with Dirichlet boundary condition (2.2). Let \(x_0 \in \partial M\) and for \(\lambda > 0\), let\[ u_{\lambda}(x, t) = u \left( \exp_{x_0} \lambda x, T + \lambda^2 t \right). \]

Let \(R > 0\) be a number less than the injectivity radius on \(M\). Using a local chart, we can identify the set \(\{ x \in M : \text{dist}(x, x_0) < R \}\) with
\[ \Omega = \{ x \in \mathbb{R}^n : |x| < R, \ x_n \geq \phi(x_1, \ldots, x_{n-1}) \}, \]
where \(\phi(x')\) is a \(C^{2,\alpha}\) function, \(\phi(0) = 0\), \(D \phi(0) = 0\). When \(0 < \lambda < 1\), \(u_{\lambda}(x, t)\) is defined on the set \(\Omega_\lambda \times (-T/\lambda, 0)\), where
\[ \Omega_\lambda = \{ (x, t) : |x| < R/\lambda, \ \lambda x_n \geq \phi(\lambda x_1, \ldots, \lambda x_{n-1}) \} \]

For each \(\lambda > 0\), we have
\[ |D u_{\lambda}|^2 (x, t) = \lambda^2 |D u|^2 (\lambda x, \lambda^2 t) \leq \frac{C_1}{|t|}. \]

By Corollary 3.2,
\[ \|u_{\lambda}(x, t)\|_{C^{2+\alpha,1+\alpha/2}(\Omega_\lambda \times (-R/\lambda, 0))} \leq \frac{C_1}{|t|}. \]

Hence, there is a subsequence \(\{ u_{\lambda_i} \}\) such that on each compact set in \(\mathbb{R}^n_+ \times (-\infty, 0)\), \(\{ u_{\lambda_i} \}\) converges in \(C^{2+\alpha,1+\alpha/2}\) to a map.
\( \bar{u} : \mathbb{R}_+^n \times (-\infty, 0) \to N \)

where \( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_n \geq 0 \} \), and \( \bar{u} \) satisfies the harmonic map heat flow. Since the function \( h \) in (2.2) is \( C^{2,\alpha} \), we have \( \bar{u}(x) = h(x_0) \) whenever \( x_n = 0 \). We claim that the function \( \bar{u} \) satisfies the dilation-invariant condition:

\[(4.3) \quad \text{for any } \lambda > 0, \quad \bar{u}(x, t) = \bar{u}(\lambda x, \lambda^2 t). \]

In fact, from the monotonicity formula Theorem 2.1, we have

\[(4.4) \quad \int_{T-1}^{T} (T-t) \int_M \left( u_t + \frac{D_{\bar{u}} \cdot Dr^2}{4(t-T)} \right)^2 G \, dx \, dt \leq C < \infty, \]

where

\[ G(x, t) = \left( \frac{1}{|T-t|} \right)^{n/2} \exp \left( \frac{\text{dist}^2(x, x_0)}{4(t-T)} \right). \]

Then, for any \( \epsilon > 0 \), we can find \( \delta > 0 \) such that

\[ \int_{T-\delta}^{T} (T-t) \int_M \left( u_t + \frac{D_{\bar{u}} \cdot Dr^2}{4(t-T)} \right)^2 G \, dx \, dt \leq \epsilon. \]

Let \( R > 0 \) be a number less than the injectivity radius on \( M \). From (4.4), for any \( \lambda > 0 \),

\[ \int_{-\delta/\lambda^2}^{0} |t| \int_{B(R/\lambda)} \left( u_{\lambda t} + \frac{D_{\bar{u}_\lambda} \cdot Dr^2}{4t} \right)^2 G_{\lambda} \, dx \, dt \leq \epsilon, \]

where

\[ G_{\lambda}(x, t) = \left( \frac{1}{\pi|t|} \right)^{n/2} \exp \left( \frac{\text{dist}^2_M(\text{exp}_{x_0}(\lambda x), x_0)}{4\lambda^2 t} \right). \]

When \( \lambda \to 0 \), we have

\[ \int_{-\infty}^{0} |t| \int_{\mathbb{R}^n} \left( \bar{u}_t + \frac{D\bar{u} \cdot x}{2t} \right)^2 \bar{G} \, dx \, dt \leq \epsilon, \]

where

\[ \bar{G}(x, t) = \left( \frac{1}{4\pi|t|} \right)^{n/2} \exp \left( \frac{|x|^2}{4t} \right) \]

is the backward heat kernel on \( \mathbb{R}^n \). Since \( \epsilon \) can be any positive number, we have

\[ \int_{-\infty}^{0} |t| \int_{\mathbb{R}_+^n} \left( \bar{u}_t + \frac{D\bar{u} \cdot x}{2t} \right)^2 \bar{G} \, dx \, dt = 0. \]

It shows that

\[ \bar{u}_t + \frac{D\bar{u} \cdot x}{2t} = 0 \quad \text{in} \quad \mathbb{R}_+^n \times (-\infty, 0), \]

and (4.3) holds.
By (4.2), when \( \lambda \to 0 \), we have
\[
|D\bar{u}|^2(x, t) \leq \frac{C_1}{|t|}.
\]

By the small energy regularity, Theorem 3.4, if \( x_0 \in \partial M \) and \((x_0, T)\) is a singular point, then, there is \( \beta > 0 \) such that for all \( T - \beta \leq t \leq T \), we have
\[
|T - t| \int_M |Du|^2(x, t)G(x, t) \, dx > \epsilon.
\]

Let \( \rho > 0 \) be large enough so that
\[
\int_{\text{dist}(x, x_0) \geq \rho \sqrt{t - T}} G(x, t) \, dx \leq \frac{\epsilon}{2C_1}.
\]
Then, for all \( T - \beta \leq t \leq T \), we have
\[
|T - t| \int_{\text{dist}(x, x_0) \leq \rho \sqrt{T - t}} |Du|^2(x, t)G(x, t) \, dx \geq \epsilon/2.
\]
Since \( u_\lambda \) converges to \( \bar{u} \) on compact sets in \( \mathbb{R}^n_+ \times (-\infty, 0) \), it is not difficult to see that the same will hold for \( \bar{u} \): for \( t < 0 \),
\[
|\epsilon| \int_{\{x \in \mathbb{R}^n_+, |x| \leq \rho \sqrt{|\epsilon|}\}} |D\bar{u}|^2(x, t)\bar{G}(x, t) \, dx \geq \epsilon/2.
\]
This implies that \( \bar{u} \) is not a constant function.

5. HARMONIC HEAT MAPS WITH NEUMANN BOUNDARY CONDITION

We say \( \partial M \) is convex, if for any \( a \in \bar{M} \),
\[
Dr \cdot \nu \geq 0 \quad \text{on} \quad \partial M
\]
where \( r(x) = \text{dist}(a, x) \) and \( \nu \) is the unit outward normal on \( \partial M \).

Suppose that \( \partial M \) is convex. Let \( u(x, t) : M \times (0, T) \to N \) be a smooth solution of the harmonic heat equation with Neumann boundary condition. Suppose that
\[
\max_{x \in M} |Du|(x, t) \to \infty \quad \text{as} \quad t \to T.
\]
As before, for any \( x_0 \in M \), let
\[
\mathcal{E}(x_0; t) = (T - t) \int_M |Du|^2(x, t)G(x_0, T; x, t) \, dx,
\]
where
\[
G(x_0, T; x, t) = \left( \frac{1}{4\pi|T - t|} \right)^{n/2} \exp \left( \frac{r^2(x_0; x)}{4(t - T)} \right).
\]
Theorem 5.1. Suppose that $\partial M$ is convex. Let $u(x,t): M \times (0, T) \to N$ be a smooth solution of the harmonic heat equation with Neumann boundary condition,

\begin{equation}
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial M \times (0, T)
\end{equation}

and

\begin{equation}
\int_M |Du|^2(x,t) \, dx \leq E_0 \quad \text{for} \quad t \in (0, T).
\end{equation}

Then there is a constant $B > 0$, depending only on $M$, $N$, $T$ and $E_0$ only, so that, for all $t \in (0, T)$,

\begin{equation}
\frac{d}{dt} \left( \exp \left(2|T-t|^{1/2}\right) \mathcal{E}(x_0; t) + B|T-t|^{1/2} \right) 
\leq -2 \exp \left(2|T-t|^{1/2}\right) |T-t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4(t-T)} \right)^2 G(x_0, T; x, t) \, dx.
\end{equation}

Proof. After a translation in time, we may assume that $u$ is defined on $M \times [-T, 0)$. As in section 2, we will write $r(x) = r(x_0; x) = \text{dist}(x_0, x)$, and

\begin{equation}
\mathcal{E}(t) = \mathcal{E}(x_0; t) = |t| \int_M |Du|^2(x,t)G(x,t) \, dx,
\end{equation}

where

\begin{equation}
G(x, t) = \left( \frac{1}{4\pi|t|} \right)^{n/2} \exp \left( \frac{r^2(x)}{4t} \right),
\end{equation}

for $x \in M$ and $t \in (-T, 0)$. By (5.1) and (5.2), equation (2.10) becomes

\begin{equation}
\mathcal{E}'(t) \leq -2|t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4t} \right)^2 G \, dx + |t| \int_M |Du|^2 (G_t + \Delta G) \, dx 

- \int_M |Du|^2(x,t)G(x,t) \, dx - 2|t| \int_M \frac{Du \cdot D^2r \cdot Du}{4t} G \, dx.
\end{equation}

By (2.13) and (2.14), we have

\begin{equation}
\mathcal{E}'(t) \leq -2|t| \int_M \left( u_t + \frac{Du \cdot Dr^2}{4t} \right)^2 G \, dx 

+ C_3|t| \int_M |Du|^2(x,t) \frac{r^2}{|t|} G(x,t) \, dx.
\end{equation}

The rest of the proof is the same as the proof of Theorem 2.1.
Lemma 5.2. Let \( u : M \times [-1,0] \rightarrow N \) be a regular solution of (2.1) with Neumann boundary condition (5.1). Suppose that for some \( A > 0 \),

\[(5.6) \quad |Du|^2(x,t) \leq A \quad \text{on} \quad P(2R).\]

Then,

\[\|u\|_{C^{2+\alpha,1+\alpha/2}(M \times (-1/8,0))} \leq CA.\]

Proof. Suppose that \( x_0 \in \partial M \). Let \( R > 0 \) be a number less than the injectivity radius of \( M \). By choosing a \( C^{2,\alpha} \) chart, we may identify a set \( \Omega \subset \{ x \in M : \text{dist}(x,x_0) < R \} \) with the set

\[D_+(R/2) = \{ x \in \mathbb{R}^n : |x| < R/2, \; x_n > 0 \}.\]

If \( R \) is chosen small enough, the map \((y_1, y_2, ..., y_n)\) is \( C^{2,\alpha} \) and its inverse exists and is \( C^{2,\alpha} \). In \( D_+(R/2) \), \( u \) is a solution of an equation of the form:

\[(5.7) \quad u_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( a^{ij} \frac{\partial u}{\partial x_j} \right) + \Gamma(Du,Du),\]

where \( a^{ij} \) and \( \Gamma \) are \( C^\alpha \) functions and \( \Gamma(Du,Du) \leq C|Du|^2 \), and

\[\frac{\partial u}{\partial x_n} = 0 \quad \text{whenever} \quad x_n = 0.\]

Let \( u(x,t) = u(-x,t) \) when \( x_n < 0 \). Then, \( u(x,t) \) is a solution of (5.7) in \( D(R/2) \times (0,T) \), where \( D(R/2) = \{ x \in \mathbb{R}^n : |x| < R/2 \} \). As in section 3, using the regularity theory and Sobolev inequality, we obtain

\[\|u\|_{C^{2+\alpha,1+\alpha/2}(B(x_0,R/8) \times (-R/8,0))} \leq CA.\]

If \( x_0 \) lies in the interior of \( M \), we argue as in Lemma 3.1. This proves the Lemma.

As in section 3, we have the small-energy-regularity result:

Theorem 5.3. Suppose that \( M \) is a compact manifold with convex boundary. There are constants \( \epsilon_4 > 0 \) and \( \beta > 0 \), depending only on \( M, \; N \) and \( h \) only, such that for any regular solution \( u : M \times [-T,0) \rightarrow N \) of (2.1) with Neumann boundary condition (5.2) and

\[
\int_M |Du|^2(x,t) \; dx \leq E_0 < \infty, \quad \text{for} \quad t \in [-T,0),
\]

the following is true: If
for some $t_0 \in (-\beta, 0)$, then there are constants $\delta > 0$, depending on $M$, $N$, $E_0$, and $\beta$ only, and $C > 0$ depending on $M$, $N$ only, so that
\[
\sup_{P(\delta \sqrt{|t_0|})} |Du|^2 \leq \frac{C}{\delta^2 |t_0|}.
\]

From Theorem 5.3, we have

**Theorem 5.4.** Suppose that $M$ is a compact manifold with strictly convex boundary. Let $u : M \times [0, T) \to N$ be a regular solution of (2.1) with Neumann boundary condition (5.2) and
\[
\int_M |Du|^2(x, t) \, dx \leq E_0 < \infty, \quad \text{for } t \in [0, T).
\]

Let $n$ be the dimension of $M$. Then, there exists a closed set $S$ with finite $n - 2$ dimensional measure such that $u(x, t)$ converges smoothly to a limit $u(x, T)$ as $t \to T$ on compact sets in $M - S$. Moreover, there exists a constant $C > 0$ depending only on $M$, $N$, $h$ and $E_0$ such that if $U$ is any relatively open set containing $S$, then
\[
\mathcal{H}_{n-2}(S) \leq C \liminf_{t \to T} \int_M |Du|^2(x, t) \, dx.
\]

Now, suppose that
\[
\sup_M |Du|^2(x, t) \leq \frac{C}{T - t}.
\]

As in section 4, we let $u(\lambda x, t) = u(\exp_{x_0} \lambda x, T + \lambda^2 t)$.

Using the almost the same arguments, we can show that if $x_0 \in M$ is a singular point, there is a sequence $\lambda_i$ such that on each compact set in $\mathbb{R}^n \times (-\infty, 0)$, $\{u_{\lambda_i}\}$ in $C^{2, \alpha}$ converges to a non-constant map
\[
\bar{u} : \mathbb{R}^n \times (-\infty, 0) \to N
\]
and $\bar{u}$ satisfies the harmonic map heat flow, and is dilation-invariant, i.e., for any $\lambda > 0$, we have
\[
\bar{u}(x, t) = \bar{u}(\lambda x, \lambda^2 t).
\]

If $x_0 \in \partial M$ is a singular point, there is a sequence $\lambda_i$ such that on each compact set in $\mathbb{R}^n_+ \times (-\infty, 0)$, $\{u_{\lambda_i}\}$ in $C^{2, \alpha}$ converges to a non-constant map
\[
\bar{u} : \mathbb{R}^n_+ \times (-\infty, 0) \to N
\]
where $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_n \geq 0 \}$, and $\bar{u}$ satisfies the harmonic map heat flow, and
\[
\frac{\partial \bar{u}}{\partial x_n}(x, t) = 0 \quad \text{whenever} \quad x_n = 0,
\]
and is dilation-invariant, i.e., for any $\lambda > 0$, we have
\[
\bar{u}(x, t) = \bar{u}(\lambda x, \lambda^2 t).
\]

REFERENCES


