A NOTE ON INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR LOG-CONVEX AND LOG-CONCAVE FUNCTIONS

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Abstract. In this note, we establish new inequalities of Hadamard type involving several log-convex functions and log-concave functions.

1. INTRODUCTION

The following integral inequality

\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \]

where \( f : [a, b] \to \mathbb{R} \) is a convex function with \( a < b \) is well known in the literature as the Hadamard inequality (see [4]). A function \( f : I \to (0, \infty) \), where \( I \) is an interval in \( \mathbb{R} \), is said to be log-convex function, if for all \( x, y \in I \) and \( t \in [0, 1] \) one has the inequality (see [6, p. 3]):

\[ f(tx + (1 - t)y) \leq [f(x)]^t[f(y)]^{1-t}, \]

\( f \) is said to be log-concave if

\[ f(tx + (1 - t)y) \geq [f(x)]^t[f(y)]^{1-t}. \]

Recall that the extended logarithmic mean \( L_p \) of two positive numbers \( a, b \) is given for \( a = b \) by \( L_p(a, a) = a \) and for \( a \neq b \) by

\[
L_p(a, b) = \begin{cases} 
\left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}, & p \neq -1, 0 \\
\frac{b - a}{\ln b - \ln a}, & p = -1 \\
\frac{1}{e} \left( \frac{b}{a} \right)^{\frac{1}{e(a-b)}}, & p = 0
\end{cases}
\]

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where the $L_{-1}(a, b)$ is the logarithmic mean $L(a, b)$. In [2], Dragomir and Mond proved that the following inequalities hold for log-convex function $f$:

$$f\left(\frac{a + b}{2}\right) \leq \exp\left[\frac{1}{b - a} \int_a^b \ln[f(x)]dx\right]$$

$$\leq \frac{1}{b - a} \int_a^b G(f(x), f(a + b - x))dx$$

$$\leq \frac{1}{b - a} \int_a^b f(x)dx \leq L(f(a), f(b))$$

$$\leq \frac{f(a) + f(b)}{2},$$

where

$$G(p, g) = \sqrt{pq},$$

is the geometric mean. For the further refinements of (1.1) for log-convex functions and various other results related to (1.1), see [1 - 3] and [5 - 7]. In [6] Pachpatte proved the following inequalities involving two log-convex functions:

**Theorem 1.1.** Let $f, g : I \rightarrow (0, \infty)$ be log-convex functions on $I$ and $a, b \in I$ with $a < b$. Then

$$\frac{2}{b - a} \int_a^b f(x)g(x)dx$$

$$\leq \frac{f(a) + f(b)}{2} L(f(a), f(b)) + \frac{g(a) + g(b)}{2} L(g(a), g(b)).$$

**Theorem 1.2.** Let $f, g : I \rightarrow (0, \infty)$ be differentiable log-convex functions on the interval of real numbers $I^0$ (the interior of $I$) and $a, b \in I^0$ with $a < b$. Then

$$\frac{2}{b - a} \int_a^b f(x)g(x)dx$$

$$\geq \frac{1}{b - a} f\left(\frac{a + b}{2}\right) \int_a^b g(x) \exp \left[\frac{f'(\frac{a + b}{2})}{f(\frac{a + b}{2})} \left(x - \frac{a + b}{2}\right)\right] dx$$

$$+ \frac{1}{b - a} g\left(\frac{a + b}{2}\right) \int_a^b f(x) \exp \left[\frac{g'(\frac{a + b}{2})}{g(\frac{a + b}{2})} \left(x - \frac{a + b}{2}\right)\right] dx.$$

The main purpose of this note is to establish some generalizations of Theorems 1.1 and 1.2 as well as some new inequalities involving several log-convex functions and log-concave functions.
2. MAIN RESULTS

**Theorem 2.1.** Let $f$, $g$, $a$, $b$ be as in Theorem 1.1 and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \alpha \left[ L_{\frac{1}{\alpha}}(f(a), f(b)) \right]^{\frac{1}{\alpha}} L(f(a), f(b)) + \beta \left[ L_{\frac{1}{\beta}}(g(a), g(b)) \right]^{\frac{1}{\beta}} L(g(a), g(b)).
\]

**Proof.** Since $f$, $g$ are log-convex functions, we have

\[
f(ta + (1-t)b) \leq [f(a)]^t[f(b)]^{1-t},
\]

\[
g(ta + (1-t)b) \leq [g(a)]^t[g(b)]^{1-t},
\]

for all $t \in [0, 1]$. It is easy to observe that

\[
\int_a^b f(x)g(x)dx = (b-a) \int_0^1 f(ta + (1-t)b)g(ta + (1-t)b)dt.
\]

Using the known inequality $cd \leq \alpha c^\frac{1}{\alpha} + \beta d^\frac{1}{\beta}$ ($\alpha, \beta > 0$ and $\alpha + \beta = 1$), (2.2), (2.3) on the right side of (2.4) and making the change of variable we have

\[
\int_a^b f(x)g(x)dx \\
\leq (b-a) \int_0^1 \left\{ \alpha [f(ta + (1-t)b)]^{\frac{1}{\alpha}} + \beta [g(ta + (1-t)b)]^{\frac{1}{\beta}} \right\} dt
\]

\[
= (b-a) \int_0^1 \left\{ \alpha \left[ \frac{f(a)}{f(b)} \right]^{-1} \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^\sigma dt + \beta \left[ \frac{g(a)}{g(b)} \right]^{-1} \int_0^1 \left[ \frac{g(a)}{g(b)} \right]^{-\sigma} d\sigma \right\}
\]

\[
= (b-a) \int_0^1 \left\{ \alpha^{\frac{1}{\alpha}} f^{-\frac{1}{\alpha}}(b) \int_0^1 \left[ \frac{f(a)}{f(b)} \right]^\sigma dt + \beta g^{-\frac{1}{\beta}}(b) \int_0^1 \left[ \frac{g(a)}{g(b)} \right]^{-\sigma} d\sigma \right\}
\]

\[
= (b-a) \left\{ \alpha^{\frac{1}{\alpha}} f^{-\frac{1}{\alpha}}(b) \log \left[ \frac{f(a)}{f(b)} \right] + \beta g^{-\frac{1}{\beta}}(b) \log \left[ \frac{g(a)}{g(b)} \right] \right\}
\]

\[
= (b-a) \left[ \alpha^{\frac{1}{\alpha}} \left( \frac{f^{-\frac{1}{\alpha}}(a) - f^{-\frac{1}{\alpha}}(b)}{\log f(a) - \log f(b)} \right) + \beta g^{-\frac{1}{\beta}}(b) \log \frac{g(a)}{g(b)} \right]
\]
Then the following inequality holds:

$$
= (b - a) \left\{ \alpha^2 \left( \frac{f^{1/\alpha}(a) - f^{1/\alpha}(b)}{f(a) - f(b)} \right) L(f(a), f(b)) \\
+ \beta^2 \left( \frac{g^{1/\beta}(a) - g^{1/\beta}(b)}{g(a) - g(b)} \right) L(g(a), g(b)) \right\}
$$

$$
= (b - a) \left\{ \alpha \left[ L_{\frac{1}{\alpha}}(f(a), f(b)) \right]^{1-\alpha} L(f(a), f(b)) \\
+ \beta \left[ L_{\frac{1}{\beta}}(g(a), g(b)) \right]^{1-\beta} L(g(a), g(b)) \right\}.
$$

Rewriting (2.5) we get the required inequality in (2.1). The proof is completed.

**Remark 2.1.** For $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$, the inequality (2.1) reduces to (1.2).

**Theorem 2.2.** Let $f, g : I \to (0, \infty)$ be log-concave functions on $I$ and $a, b \in I$ with $a < b$. Further, let $\alpha > 1$ with $\alpha + \beta = 1$ (or $\beta > 1$ with $\alpha + \beta = 1$).

Then the following inequality holds:

$$
\frac{1}{b - a} \int_a^b f(x) g(x) dx \geq \alpha \left[ L_{\frac{1}{\alpha}}(f(a), f(b)) \right]^{1-\alpha} L(f(a), f(b)) \\
+ \beta \left[ L_{\frac{1}{\beta}}(g(a), g(b)) \right]^{1-\beta} L(g(a), g(b)).
$$

**Proof.** Since $f, g$ are log-concave functions, we have

$$
f(ta + (1 - t)b) \geq \frac{(f)(a)^t [f(b)]^{1-t}},
$$

$$
g(ta + (1 - t)b) \geq \frac{(g)(a)^t [g(b)]^{1-t}},
$$

for all $t \in [0, 1]$. Using the known inequality $cd \geq \alpha c^{1/\alpha} + \beta d^{1/\beta}$, (2.7), (2.8) on the right side of (2.4) and making the change of variable we have

$$
\int_a^b f(x) g(x) dx \\
\geq (b - a) \int_0^1 \left\{ \alpha \left[ f(ta + (1 - t)b) \right]^{1/\alpha} + \beta \left[ g(ta + (1 - t)b) \right]^{1/\beta} \right\} dt
$$

$$
\geq (b - a) \int_0^1 \left\{ \alpha \left[ (f(a))^t [f(b)]^{1-t} \right]^{1/\alpha} + \beta \left[ (g(a))^t [g(b)]^{1-t} \right]^{1/\beta} \right\} dt
$$

$$
= (b - a) \left\{ \alpha \left[ L_{\frac{1}{\alpha}}(f(a), f(b)) \right]^{1-\alpha} L(f(a), f(b)) \\
+ \beta \left[ L_{\frac{1}{\beta}}(g(a), g(b)) \right]^{1-\beta} L(g(a), g(b)) \right\}.
$$
Rewriting (2.9) we get the required inequality in (2.6). The proof is completed.

**Theorem 2.3.** Let \( f, g, a, b \) be as in Theorem 1.2 and \( \alpha, \beta > 0 \) with \( \alpha + \beta = 1 \). Then the following inequality holds:

\[
\int_a^b f(x)g(x)dx \geq \alpha f \left( \frac{a+b}{2} \right) \int_a^b g(x) \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx
\]

(2.10)

\[
+ \beta g \left( \frac{a+b}{2} \right) \int_a^b f(x) \exp \left[ \frac{g'( \frac{a+b}{2} )}{g \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx
\]

**Proof.** Since \( f, g \) are differentiable and log-convex functions on \( I^0 \), we have that

\[
\log f(x) - \log f(y) = \log \left( \frac{f(x)}{f(y)} \right) \geq \frac{f'(y)}{f(y)}(x - y),
\]

(2.11)

\[
\log g(x) - \log g(y) = \log \left( \frac{g(x)}{g(y)} \right) \geq \frac{g'(y)}{g(y)}(x - y),
\]

(2.12)

for all \( x, y \in I^0 \). That is

\[
f(x) \geq f(y) \exp \left[ \frac{f'(y)}{f(y)}(x - y) \right],
\]

(2.13)

\[
g(x) \geq g(y) \exp \left[ \frac{g'(y)}{g(y)}(x - y) \right],
\]

(2.14)

Multiplying both sides of (2.13) and (2.14) by \( \alpha g(x) \) and \( \beta f(x) \) respectively and adding the resulting inequalities we have

\[
f(x)g(x) \geq \alpha g(x)f \left( \frac{a+b}{2} \right) \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right]
\]

(2.15)

\[+ \beta f(x)g \left( \frac{a+b}{2} \right) \exp \left[ \frac{g'( \frac{a+b}{2} )}{g \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right].
\]

By taking \( y = \frac{a+b}{2} \) in (2.15) we have

\[
f(x)g(x) \geq \alpha g(x)f \left( \frac{a+b}{2} \right) \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right]
\]

(2.16)

\[+ \beta f(x)g \left( \frac{a+b}{2} \right) \exp \left[ \frac{g'( \frac{a+b}{2} )}{g \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right].
\]
Integrating both sides of (2.14) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality (2.10).

**Remark 2.2.** For \( \alpha = \frac{1}{2}, \beta = \frac{1}{2} \), the inequality (2.6) reduces to the inequality (1.3).

**Theorem 2.4.** Let \( f, g : I \rightarrow (0, \infty) \) be differentiable log-concave functions on the interval of real numbers \( I^0 \) and \( a, b, \alpha, \beta \) be as in Theorem 2.3. Then the following inequality holds:

\[
\int_a^b f(x)g(x)dx \leq \alpha f \left( \frac{a+b}{2} \right) \int_a^b g(x) \exp \left[ \frac{f' \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx + \beta g \left( \frac{a+b}{2} \right) \int_a^b f(x) \exp \left[ \frac{g' \left( \frac{a+b}{2} \right)}{g \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx.
\]

**(2.17)**

**Proof.** Since \( f, g \) are differentiable and log-concave functions on \( I^0 \), we have that

\[
\log f(x) - \log f(y) = \log \left( \frac{f(x)}{f(y)} \right) \leq \frac{f'(y)}{f(y)} (x - y),
\]

**(2.18)**

\[
\log g(x) - \log g(y) = \log \left( \frac{g(x)}{g(y)} \right) \leq \frac{g'(y)}{g(y)} (x - y),
\]

**(2.19)**

for all \( x, y \in I^0 \). That is

\[
f(x) \leq f(y) \exp \left[ \frac{f'(y)}{f(y)} (x - y) \right],
\]

**(2.20)**

\[
g(x) \leq g(y) \exp \left[ \frac{g'(y)}{g(y)} (x - y) \right].
\]

**(2.21)**

Multiplying both sides of (2.20) and (2.21) by \( \alpha g(x) \) and \( \beta f(x) \) respectively and adding the resulting inequalities we have

\[
f(x)g(x) \leq \alpha g(x)f(y) \exp \left[ \frac{f'(y)}{f(y)} (x - y) \right] + \beta f(x)g(y) \exp \left[ \frac{g'(y)}{g(y)} (x - y) \right].
\]

**(2.22)**

By taking \( y = \frac{a+b}{2} \) in (2.22) we have

\[
f(x)g(x) \leq \alpha g(x)f \left( \frac{a+b}{2} \right) \exp \left[ \frac{f' \left( \frac{a+b}{2} \right)}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] + \beta f(x)g \left( \frac{a+b}{2} \right) \exp \left[ \frac{g' \left( \frac{a+b}{2} \right)}{g \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right].
\]

**(2.23)**
Integrating both sides of (2.23) with respect to $x$ from $a$ to $b$, we get the desired inequality (2.17).

**Theorem 2.5.** Let $f, a, b$ be as in Theorem 1.2 and $g$ be as in Theorem 2.4. Further, let $\alpha > 1$ with $\alpha + \beta = 1$, then the following inequality holds:

\[
\int_a^b f(x)g(x)dx \\
\geq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[\frac{f'(\frac{a+b}{2})}{f\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2}\right)\right] dx \\
+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[\frac{g'(\frac{a+b}{2})}{g\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2}\right)\right] dx.
\]

**Proof.** Since $f$ is differentiable and log-convex functions on $I^0$ and $g$ is differentiable and log-concave functions on $I^0$, we have that

\[
f(x) \geq f(y) \exp \left[\frac{f'(y)}{f(y)}(x - y)\right],
\]

\[
g(x) \leq g(y) \exp \left[\frac{g'(y)}{g(y)}(x - y)\right].
\]

Multiplying both sides of (2.25) and (2.26) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we have

\[
f(x)g(x) \\
\geq \alpha g(x)f(y) \exp \left[f'(y)\left(x - y\right)\right] + \beta f(x)g(y) \exp \left[g'(y)\left(x - y\right)\right].
\]

By taking $y = \frac{a+b}{2}$ in (2.27) we have

\[
f(x)g(x) \geq \alpha g(x)f\left(\frac{a+b}{2}\right) \exp \left[\frac{f'(\frac{a+b}{2})}{f\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2}\right)\right] \\
+ \beta f(x)g\left(\frac{a+b}{2}\right) \exp \left[\frac{g'(\frac{a+b}{2})}{g\left(\frac{a+b}{2}\right)} \left(x - \frac{a+b}{2}\right)\right].
\]

Integrating both sides of (2.28) with respect to $x$ from $a$ to $b$, we get the desired inequality (2.24).

**Theorem 2.6.** Let $g, a, b$ be as in Theorem 1.2 and $f$ be as in Theorem 2.4.
Further, let \( \alpha > 1 \) with \( \alpha + \beta = 1 \), then the following inequality holds:

\[
\int_a^b f(x)g(x)dx \\
\leq \alpha f\left(\frac{a+b}{2}\right) \int_a^b g(x) \exp \left[ \frac{f'(\frac{a+b}{2})}{f\left(\frac{a+b}{2}\right)} \left( x - \frac{a+b}{2} \right) \right] dx \\
+ \beta g\left(\frac{a+b}{2}\right) \int_a^b f(x) \exp \left[ \frac{g'(\frac{a+b}{2})}{g\left(\frac{a+b}{2}\right)} \left( x - \frac{a+b}{2} \right) \right] dx.
\]

(2.29)

**Proof.** Multiplying both sides of (2.20) and (2.14) by \( \alpha g(x) \) and \( \beta f(x) \) respectively and adding the resulting inequalities we have

\[
f(x)g(x) \\
\leq \alpha g(x)f(y) \exp \left[ \frac{f'(y)}{f(y)} (x - y) \right] + \beta f(x)g(y) \exp \left[ \frac{g'(y)}{g(y)} (x - y) \right].
\]

(2.30)

By taking \( y = \frac{a+b}{2} \) in (2.30) we have

\[
f(x)g(x) \leq \alpha g(x)f\left(\frac{a+b}{2}\right) \exp \left[ \frac{f'(\frac{a+b}{2})}{f\left(\frac{a+b}{2}\right)} \left( x - \frac{a+b}{2} \right) \right] \\
+ \beta f(x)g\left(\frac{a+b}{2}\right) \exp \left[ \frac{g'(\frac{a+b}{2})}{g\left(\frac{a+b}{2}\right)} \left( x - \frac{a+b}{2} \right) \right].
\]

(2.31)

Integrating both sides of (2.31) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality (2.29).

**Theorem 2.7.** Let \( f_1, f_2, \ldots, f_n : I \to (0, \infty) \) be log-convex functions on \( I \) and \( a, b \in I \) with \( a < b \). Further, let \( \alpha_1, \alpha_2, \ldots, \alpha_n > 0 \) with \( \sum_{i=1}^n \alpha_i = 1 \). Then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b \sum_{i=1}^n f_i(x)dx \\
\leq \sum_{i=1}^n \left\{ \alpha_i \left[ L^{\frac{1}{\alpha_i}-1}(f_i(a), f_i(b)) \right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b)) \right\}.
\]

(2.32)

**Proof.** Since \( f_1, f_2, \ldots, f_n \) are log-convex functions, we have

\[
f_i(ta + (1-t)b) \leq [f_i(a)]^t [f_i(b)]^{1-t},
\]

(2.33)
for all $t \in [0, 1]$, $i = 1, 2, \cdots, n$. Since

\begin{equation}
\int_a^b \sum_{i=1}^n f_i(x)dx = (b - a) \int_0^1 \sum_{i=1}^n f_i(ta + (1 - t)b)dt.
\end{equation}

Using the inequality $f_1 f_2 \cdots f_n \leq \alpha_1 (f_1)^{\frac{1}{\alpha_1}} + \alpha_2 (f_2)^{\frac{1}{\alpha_2}} + \cdots + \alpha_n (f_n)^{\frac{1}{\alpha_n}}$ and (2.33) on the right side of (2.34) and making the change of variable we have

\begin{equation}
\begin{aligned}
\int_a^b \sum_{i=1}^n f_i(x)dx \\
&\leq (b - a) \int_0^1 \left\{ \sum_{i=1}^n \alpha_i [f_i(ta + (1 - t)b)]^{\frac{1}{\alpha_i}} \right\} dt \\
&\leq (b - a) \int_0^1 \sum_{i=1}^n \alpha_i \{[f_i(a)]^t/[f_i(b)]^{1-t}\}^{\frac{1}{\alpha_i}} dt \\
&= (b - a) \sum_{i=1}^n \alpha_i \left\{ \alpha_i f_i^{\frac{1}{\alpha_i}}(b) \int_0^1 \frac{f_i(a)}{f_i(b)}^{\frac{1}{\alpha_i}} dt \right\} \\
&= (b - a) \sum_{i=1}^n \left\{ \alpha_i^2 f_i^{\frac{1}{\alpha_i}}(b) \int_0^1 \frac{f_i(a)}{f_i(b)}^{\frac{1}{\alpha_i}} \sigma d\sigma \right\} \\
&= (b - a) \sum_{i=1}^n \left\{ \alpha_i^2 \left( \frac{f_i^{\frac{1}{\alpha_i}}(a) - f_i^{\frac{1}{\alpha_i}}(b)}{\log f_i(a) - \log f_i(b)} \right) \right\} \\
&= (b - a) \sum_{i=1}^n \left\{ \alpha_i^2 \left( \frac{f_i^{\frac{1}{\alpha_i}}(a) - f_i^{\frac{1}{\alpha_i}}(b)}{f_i(a) - f_i(b)} \right) L(f_i(a), f_i(b)) \right\} \\
&= (b - a) \sum_{i=1}^n \left\{ \alpha_i \left[ L_{\frac{1}{\alpha_i}}^{-1}(f_i(a), f_i(b)) \right]^{\frac{1}{\alpha_i}} L(f_i(a), f_i(b)) \right\}
\end{aligned}
\end{equation}

Rewriting (2.35) we get the required inequality in (2.32). The proof is completed.

**Remark 2.3.** For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, the inequality (2.32) reduces to

\begin{equation}
\frac{n}{b - a} \int_a^b \sum_{i=1}^n f_i(x)dx \\
\leq \sum_{i=1}^n [L_{\frac{1}{\alpha_i}}^{-1}(f_i(a), f_i(b))]^{n-1} L(f_i(a), f_i(b)).
\end{equation}

**Remark 2.4.** If we choose $n = 2$ in (2.36), then (2.36) reduces to (1.2).
Theorem 2.8. Let $f_1, f_2, \ldots, f_n : I \to (0, \infty)$ be log-concave functions on $I$ and $a, b \in I$ with $a < b$. Further, let $\alpha_1 > 1$ and $\alpha_2, \alpha_3, \ldots, \alpha_n < 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, and let $\sum_{i=1}^{n} \alpha_i > 0$, $j = 2, 3, \ldots, n$. Then the following inequality holds:

$$
\frac{1}{b-a} \int_{a}^{b} \sum_{i=1}^{n} f_i(x) \, dx \geq \sum_{i=1}^{n} \left\{ \alpha_i \left[ \frac{L_{\frac{1}{\alpha_i}}(f_i(a), f_i(b))}{\alpha_i} \right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b)) \right\}.
$$

(2.37)

Proof. Since $f_1, f_2, \ldots, f_n$ are log-concave functions, we have

$$
f_i(ta + (1-t)b) \geq \{f_i(a)\}^t \{f_i(b)\}^{1-t},
$$

(2.38)

for all $t \in [0, 1], i = 1, 2, \ldots, n$. Since

$$
\int_{a}^{b} \sum_{i=1}^{n} f_i(x) \, dx = (b-a) \int_{0}^{1} \sum_{i=1}^{n} f_i(ta + (1-t)b) \, dt.
$$

(2.39)

Using the inequality $f_1 f_2 \cdots f_n \geq \alpha_1 (f_1)^{\frac{1}{\alpha_1}} + \alpha_2 (f_2)^{\frac{1}{\alpha_2}} + \cdots + \alpha_n (f_n)^{\frac{1}{\alpha_n}}$ and (2.38) on the right side of (2.39) and making the change of variable we have

$$
\int_{a}^{b} \sum_{i=1}^{n} f_i(x) \, dx \geq (b-a) \int_{0}^{1} \left\{ \sum_{i=1}^{n} \alpha_i \left[ \{f_i(a)\}^t \{f_i(b)\}^{1-t} \right]^{\frac{1-\alpha_i}{\alpha_i}} \right\} \, dt
$$

(2.40)

$$
\geq (b-a) \int_{0}^{1} \left\{ \sum_{i=1}^{n} \alpha_i \left[ \frac{L_{\frac{1}{\alpha_i}}(f_i(a), f_i(b))}{\alpha_i} \right]^{\frac{1-\alpha_i}{\alpha_i}} \right\} \, dt
$$

$$
= (b-a) \sum_{i=1}^{n} \left\{ \alpha_i \left[ \frac{L_{\frac{1}{\alpha_i}}(f_i(a), f_i(b))}{\alpha_i} \right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b)) \right\}.
$$

Rewriting (2.40) we get the required inequality in (2.37). The proof is completed.

Theorem 2.9. Let $f, g$ and $h : I \to (0, \infty)$ be differentiable log-convex functions on the interval of real numbers $I^0$ and $a, b \in I^0$ with $a < b$. Then the following
inequality holds:

\[
3 \int_a^b f(x)g(x)h(x)dx 
\geq f \left( \frac{a + b}{2} \right) \int_a^b g(x)h(x) \exp \left[ \frac{f'(\frac{a + b}{2})}{f(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right] dx 
\]

(2.41)

\[
+ g \left( \frac{a + b}{2} \right) \int_a^b f(x)h(x) \exp \left[ \frac{g'(\frac{a + b}{2})}{g(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right] dx 
\]

\[
+ h \left( \frac{a + b}{2} \right) \int_a^b f(x)g(x) \exp \left[ \frac{h'(\frac{a + b}{2})}{h(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right] dx.
\]

Proof. Since \( f, g \) and \( h \) are differentiable and log-convex functions on \( I^0 \), we have that

(2.42) \[ f(x) \geq f(y) \exp \left[ \frac{f'(y)}{f(y)} (x - y) \right], \]

(2.43) \[ g(x) \geq g(y) \exp \left[ \frac{g'(y)}{g(y)} (x - y) \right], \]

(2.44) \[ h(x) \geq h(y) \exp \left[ \frac{h'(y)}{h(y)} (x - y) \right], \]

for all \( x, y \in I^0 \). Multiplying both sides of (2.42), (2.43) and (2.44) by \( g(x)h(x) \), \( f(x)h(x) \) and \( f(x)g(x) \) respectively and adding the resulting inequalities we have

(2.45) \[
3f(x)g(x)h(x) \geq g(x)h(x)f(y) \exp \left[ \frac{f'(y)}{f(y)} (x - y) \right] 
\]

\[
+ f(x)h(x)g(y) \exp \left[ \frac{g'(y)}{g(y)} (x - y) \right] 
\]

\[
+ f(x)g(x)h(y) \exp \left[ \frac{h'(y)}{h(y)} (x - y) \right].
\]

Now, if we choose \( y = \frac{a + b}{2} \), from (2.45) we obtain

(2.46) \[
3f(x)g(x)h(x) \geq g(x)h(x)f \left( \frac{a + b}{2} \right) \exp \left[ \frac{f'(\frac{a + b}{2})}{f(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right] 
\]

\[
+ f(x)h(x)g \left( \frac{a + b}{2} \right) \exp \left[ \frac{g'(\frac{a + b}{2})}{g(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right] 
\]

\[
+ f(x)g(x)h \left( \frac{a + b}{2} \right) \exp \left[ \frac{h'(\frac{a + b}{2})}{h(\frac{a + b}{2})} \left( x - \frac{a + b}{2} \right) \right].
\]
Integrating both sides of (2.46) with respect to $x$ from $a$ to $b$, we get the desired inequality (2.41). The proof is completed.

**Remark 2.5.** For $h(x) \equiv 1$, the inequality (2.41) is reduces to (1.3).

**Remark 2.6.** Since $\frac{e^x - e^{-x}}{2x} > 1$ for $x > 0$, it follows that if we choose $g(x) = h(x) \equiv 1$ in (2.41), we have

$$
\int_a^b f(x) dx \geq f \left( \frac{a+b}{2} \right) \int_a^b \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx
$$

$$
= f \left( \frac{a+b}{2} \right) \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( b-a \right) \right] - \exp \left[ \frac{f'( \frac{a+b}{2} )}{f \left( \frac{a+b}{2} \right)} \left( b-a \right) \right] (b-a)
$$

$$
> f \left( \frac{a+b}{2} \right) (b-a)
$$

which is the first part of the inequality (1.1).

**Theorem 2.10.** Let $f_1, f_2, \cdots, f_n : I \to (0, \infty)$ be differentiable log-convex functions on the interval of real numbers $I^0$ and $a, b \in I^0$ with $a < b$. Further, let $\alpha_1, \alpha_2, \cdots, \alpha_n > 0$ with $\sum_{i=1}^n \alpha_i = 1$. Then the following inequality holds:

$$
\int_a^b \sum_{i=1}^n f_i(x) dx
$$

$$
\geq \alpha_1 f_1 \left( \frac{a+b}{2} \right) \int_a^b f_2(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f'_1 \left( \frac{a+b}{2} \right)}{f_1 \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx
$$

$$
+ \alpha_2 f_2 \left( \frac{a+b}{2} \right) \int_a^b f_1(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f'_2 \left( \frac{a+b}{2} \right)}{f_2 \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx
$$

$$
+ \alpha_n f_n \left( \frac{a+b}{2} \right) \int_a^b f_1(x) \cdots f_{n-1}(x) \exp \left[ \frac{f'_n \left( \frac{a+b}{2} \right)}{f_n \left( \frac{a+b}{2} \right)} \left( x - \frac{a+b}{2} \right) \right] dx.
$$

**Proof.** Since $f_1, f_2, \cdots, f_n$ are differentiable and log-convex functions on $I^0$, we have

$$
f_1(x) \geq f_1(y) \exp \left[ \frac{f'_1(y)}{f_1(y)} (x - y) \right],
$$

$$
f_2(x) \geq f_2(y) \exp \left[ \frac{f'_2(y)}{f_2(y)} (x - y) \right],
$$
for all $x, y \in I^0$. Multiplying both sides of (2.48 - 1), (2.48 - 2), \ldots and (2.48 - n) by $\alpha_1 f_2(x) f_3(x) \cdots f_n(x), \alpha_2 f_1(x) f_3(x) \cdots f_n(x), \cdots$ and $\alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x)$ respectively and adding the resulting inequalities we have

$$
\sum_{i=1}^{n} f_i(x) \geq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1(y) \exp \left[ \frac{f_1'(y)}{f_1(y)} (x - y) \right] \\
+ \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2(y) \exp \left[ \frac{f_2'(y)}{f_2(y)} (x - y) \right] \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(y) \exp \left[ \frac{f_n'(y)}{f_n(y)} (x - y) \right].
$$

(2.49)

Now, if we choose $y = \frac{a+b}{2}$, from (2.49) we obtain

$$
\sum_{i=1}^{n} f_i(x) \geq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1\left(\frac{a+b}{2}\right) \exp \left[ \frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right] \\
+ \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2\left(\frac{a+b}{2}\right) \exp \left[ \frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right] \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n\left(\frac{a+b}{2}\right) \exp \left[ \frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right].
$$

(2.50)

Integrating both sides of (2.50) with respect to $x$ from $a$ to $b$, we get the desired inequality (2.47). The proof is completed.

**Remark 2.7.** If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, then the inequality (2.47) reduces to

$$
n \int_{a}^{b} \sum_{i=1}^{n} f_i(x) dx \\
\geq f_1\left(\frac{a+b}{2}\right) \int_{a}^{b} f_2(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right] dx \\
+ f_2\left(\frac{a+b}{2}\right) \int_{a}^{b} f_1(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right] dx \\
\vdots \\
+ f_n\left(\frac{a+b}{2}\right) \int_{a}^{b} f_1(x) \cdots f_{n-1}(x) \exp \left[ \frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)} (x - \frac{a+b}{2}) \right] dx.
$$

(2.51)
Remark 2.8. We note that the inequality (2.41) is a special case of the inequality (2.51) when \( n = 3 \).

Theorem 2.11. Let \( f_1, f_2, \cdots, f_n : I \to (0, \infty) \) be differentiable log-concave functions on the interval of real numbers \( I^0 \) and \( a, b \in I^0 \) with \( a < b \). Further, let \( \alpha_1, \alpha_2, \cdots, \alpha_n > 0 \) with \( \sum_{i=1}^{n} \alpha_i = 1 \). Then the following inequality holds:

\[
\int_{a}^{b} \sum_{i=1}^{n} f_i(x)dx \\
\leq \alpha_1 f_1 \left( \frac{a+b}{2} \right) \int_{a}^{b} f_2(x)f_3(x)\cdots f_n(x) \exp \left[ \frac{f_1(x)}{f_1(\frac{a+b}{2})} \left( x - \frac{a+b}{2} \right) \right] dx \\
+ \alpha_2 f_2 \left( \frac{a+b}{2} \right) \int_{a}^{b} f_1(x)f_3(x)\cdots f_n(x) \exp \left[ \frac{f_2(x)}{f_2(\frac{a+b}{2})} \left( x - \frac{a+b}{2} \right) \right] dx \\
\vdots \\
+ \alpha_n f_n \left( \frac{a+b}{2} \right) \int_{a}^{b} f_1(x)\cdots f_{n-1}(x) \exp \left[ \frac{f_n(x)}{f_n(\frac{a+b}{2})} \left( x - \frac{a+b}{2} \right) \right] dx.
\]

Proof. Since \( f_1, f_2, \cdots, f_n \) are differentiable and log-concave functions on \( I^0 \), we have

\[
\begin{align*}
(2.53-1) & \quad f_1(x) \leq f_1(y) \exp \left[ \frac{f_1(y)}{f_1(y)} (x - y) \right], \\
(2.53-2) & \quad f_2(x) \leq f_2(y) \exp \left[ \frac{f_2(y)}{f_2(y)} (x - y) \right], \\
& \quad \vdots \\
(2.53-n) & \quad f_n(x) \leq f_n(y) \exp \left[ \frac{f_n(y)}{f_n(y)} (x - y) \right],
\end{align*}
\]

for all \( x, y \in I^0 \). Multiplying both sides of (2.53-1), (2.53-2), \( \cdots \) and (2.53-n) by \( \alpha_1 f_2(x)f_3(x)\cdots f_n(x), \alpha_2 f_1(x)f_3(x)\cdots f_n(x), \cdots \) and \( \alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x) \) respectively and adding the resulting inequalities we have

\[
\sum_{i=1}^{n} f_i(x) \leq \alpha_1 f_2(x)f_3(x)\cdots f_n(x)f_1(y) \exp \left[ \frac{f_1(y)}{f_1(y)} (x - y) \right] \\
+ \alpha_2 f_1(x)f_3(x)\cdots f_n(x)f_2(y) \exp \left[ \frac{f_2(y)}{f_2(y)} (x - y) \right] \\
\vdots
\]

(2.54)
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\[ + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(y) \exp \left[ \frac{f'_n(y)}{f_n(y)} (x - y) \right]. \]

Now, if we choose \( y = \frac{a + b}{2} \), from (2.54) we obtain

\[ \sum_{i=1}^{n} f_i(x) \leq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1 \left( \frac{a + b}{2} \right) \exp \left[ \frac{f'_1 \left( \frac{a + b}{2} \right)}{f_1 \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right] \]

(2.55)

\( + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2 \left( \frac{a + b}{2} \right) \exp \left[ \frac{f'_2 \left( \frac{a + b}{2} \right)}{f_2 \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right] \]

\[ \vdots \]

\[ + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n \left( \frac{a + b}{2} \right) \exp \left[ \frac{f'_n \left( \frac{a + b}{2} \right)}{f_n \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right]. \]

Integrating both sides of (2.55) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality (2.52). The proof is completed.

**Theorem 2.12.** Let \( f_1, a, b \) be as in Theorem 2.10 and \( f_2, f_3, \ldots, f_n \) be as in Theorem 2.11. Further, let \( \alpha_1 > 1, \alpha_j < 0, j = 2, 3, \ldots, n \) with \( \sum_{i=1}^{n} \alpha_i = 1 \), then the following inequality holds:

\[ \int_{a}^{b} \sum_{i=1}^{n} f_i(x) dx \]

\[ \geq \alpha_1 f_1 \left( \frac{a + b}{2} \right) \int_{a}^{b} f_2(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f'_1 \left( \frac{a + b}{2} \right)}{f_1 \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right] dx \]

(2.56)

\[ + \alpha_2 f_2 \left( \frac{a + b}{2} \right) \int_{a}^{b} f_1(x) f_3(x) \cdots f_n(x) \exp \left[ \frac{f'_2 \left( \frac{a + b}{2} \right)}{f_2 \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right] dx \]

\[ \vdots \]

\[ + \alpha_n f_n \left( \frac{a + b}{2} \right) \int_{a}^{b} f_1(x) \cdots f_{n-1}(x) \exp \left[ \frac{f'_n \left( \frac{a + b}{2} \right)}{f_n \left( \frac{a + b}{2} \right)} \left( x - \frac{a + b}{2} \right) \right] dx. \]

**Proof.** Multiplying both sides of (2.48 – 1), (2.53 – 2), \ldots and (2.53 – n) by \( \alpha_1 f_2(x) f_3(x) \cdots f_n(x) \), \( \alpha_2 f_1(x) f_3(x) \cdots f_n(x) \), \ldots and \( \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) \) respectively and adding the resulting inequalities we have

\[ \sum_{i=1}^{n} f_i(x) \geq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1(y) \exp \left[ \frac{f'_1(y)}{f_1(y)} (x - y) \right] \]

(2.57)

\[ + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2(y) \exp \left[ \frac{f'_2(y)}{f_2(y)} (x - y) \right] \]

\[ \vdots \]
\[ + \alpha_n f_1(x)f_2(x) \cdots f_{n-1}(x)f_n(y) \exp \left[ \frac{f_n'(y)}{f_n(y)} (x - y) \right]. \]

Now, if we choose \( y = \frac{a + b}{2} \), from (2.57) we obtain

\[
\sum_{i=1}^{n} f_i(x) \\
\geq \alpha_1 f_2(x)f_3(x) \cdots f_n(x)f_1 \left( \frac{a + b}{2} \right) \exp \left[ \frac{f_1' \left( \frac{a+b}{2} \right)}{f_1 \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right] \\
+ \alpha_2 f_1(x)f_3(x) \cdots f_n(x)f_2 \left( \frac{a + b}{2} \right) \exp \left[ \frac{f_2' \left( \frac{a+b}{2} \right)}{f_2 \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right] \\
\vdots \\
+ \alpha_n f_1(x)f_2(x) \cdots f_{n-1}(x)f_n \left( \frac{a + b}{2} \right) \exp \left[ \frac{f_n' \left( \frac{a+b}{2} \right)}{f_n \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right].
\]  

Integrating both sides of (2.58) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality (2.56). The proof is completed.

**Theorem 2.13.** Let \( f_2, f_3, \cdots, f_n, a, b \) be as in Theorem 2.10 and \( f_1 \) be as in Theorem 2.11. Further, let \( \alpha_1 > 1, \alpha_j < 0, j = 2, 3, \cdots, n \) with \( \sum_{i=1}^{n} \alpha_i = 1 \), then the following inequality holds:

\[
\int_{a}^{b} \sum_{i=1}^{n} f_i(x)dx \\
\leq \alpha_1 f_1 \left( \frac{a+b}{2} \right) \int_{a}^{b} f_2(x)f_3(x) \cdots f_n(x) \exp \left[ \frac{f_1' \left( \frac{a+b}{2} \right)}{f_1 \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right] dx \\
+ \alpha_2 f_2 \left( \frac{a+b}{2} \right) \int_{a}^{b} f_1(x)f_3(x) \cdots f_n(x) \exp \left[ \frac{f_2' \left( \frac{a+b}{2} \right)}{f_2 \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right] dx \\
\vdots \\
+ \alpha_n f_n \left( \frac{a+b}{2} \right) \int_{a}^{b} f_1(x) \cdots f_{n-1}(x) \exp \left[ \frac{f_n' \left( \frac{a+b}{2} \right)}{f_n \left( \frac{a+b}{2} \right)} (x - \frac{a+b}{2}) \right] dx.
\]

**Proof.** Multiplying both sides of (2.53 - 1), (2.48 - 2), \cdots and (2.48 - n) by \( \alpha_1 f_2(x)f_3(x) \cdots f_n(x), \alpha_2 f_1(x)f_3(x) \cdots f_n(x), \cdots \) and \( \alpha_n f_1(x)f_2(x) \cdots f_{n-1}(x) \) respectively and adding the resulting inequalities we have
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\[
\sum_{i=1}^{n} f_i(x) \leq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1(y) \exp \left[ \frac{f_1'(y)}{f_1(y)} (x - y) \right] \\
+ \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2(y) \exp \left[ \frac{f_2'(y)}{f_2(y)} (x - y) \right] \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(y) \exp \left[ \frac{f_n'(y)}{f_n(y)} (x - y) \right]
\]

(2.60)

Now, if we choose \( y = \frac{a+b}{2} \), from (2.60) we obtain

\[
\sum_{i=1}^{n} f_i(x) \\
\leq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1 \left( \frac{a+b}{2} \right) \exp \left[ \frac{f_1' \left( \frac{a+b}{2} \right)}{f_1 \left( \frac{a+b}{2} \right)} (x - a - \frac{b}{2}) \right] \\
+ \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2 \left( \frac{a+b}{2} \right) \exp \left[ \frac{f_2' \left( \frac{a+b}{2} \right)}{f_2 \left( \frac{a+b}{2} \right)} (x - a - \frac{b}{2}) \right] \\
\vdots \\
+ \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n \left( \frac{a+b}{2} \right) \exp \left[ \frac{f_n' \left( \frac{a+b}{2} \right)}{f_n \left( \frac{a+b}{2} \right)} (x - a - \frac{b}{2}) \right].
\]

(2.61)

Integrating both sides of (2.61) with respect to \( x \) from \( a \) to \( b \), we get the desired inequality (2.59). The proof is completed.

REFERENCES

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