Abstract. In this report, we present an order-theoretic version of the Cantor theorem. This result, which is based on the interplay of the notions of partial order and of completeness, permits to give a unified and simplified account to a long list of results related to the Bishop–Phelps theorem. We survey briefly only its simplest applications and refer the reader to [10] for a complete presentation of the results.

1. Cantor Spaces

Let \((X, \preceq)\) be a partially ordered set. For any \(z \in X\), denote the terminal tail \(\{y \in X \mid z \preceq y\}\) by \(Tz\); if \(y \in Tz\), the set \(Ty \subset Tz\) is called a subtail of \(Tz\). Clearly an element \(y\) is maximal in \((X, \preceq)\) provided \(\{y\} = Ty\). A map \(F : X \to X\) is said to be expanding if \(x \preceq F(x)\) for each \(x \in X\). We observe that if \(F : X \to X\) is expanding then: (i) any tail in \((X, \preceq)\) is invariant under \(F\), (ii) any maximal element of \((X, \preceq)\) is a fixed point of \(F\).

Let \((X; d, \preceq)\) be a metric space in which a partial order \(\preceq\) is defined. We say that \((X; d, \preceq)\) admits arbitrarily small tails if for each tail \(Tz\) and any \(\varepsilon > 0\) there exists a subtail \(Ty \subset Tz\) with \(\text{diam}(Ty) \leq \varepsilon\).

Proposition 1. Let \((X; d, \preceq)\) be a partially ordered complete metric space which admits arbitrarily small tails. Then for any \(x_0 \in X\) there exists an
ascending and convergent sequence \( x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \) such that \( \lim_{n \to \infty} x_n \in \bigcap_{n \in \mathbb{N}} T x_n \).

**Proof.** The point \( x_0 \) being given, we first choose \( x_1 \in T x_0 \) such that \( \text{diam}(T x_1) \leq 1 \). Assume that we have an ascending finite sequence \( x_0 \leq x_1 \leq \cdots \leq x_n \) such that \( \text{diam}(T x_k) \leq 1/k \) for \( 0 < k \leq n \). Choose \( x_{n+1} \in T x_n \) such that \( \text{diam}(T x_{n+1}) \leq 1/(n+1) \). By induction, we have an increasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) with \( \text{diam}(T x_n) \leq 1/n \) for each \( n > 0 \). The sequence of sets \( \{T x_n\}_{n \in \mathbb{N}} \) is clearly decreasing, so by the Cantor Theorem there exists a point \( \hat{x} \in X \) such that \( \{\hat{x}\} = \bigcap_{n \in \mathbb{N}} T x_n \). Obviously, \( \hat{x} = \lim_{n \to \infty} x_n \).

**Proposition 2.** Let \((X; d, \preceq)\) be a partially ordered complete metric space which admits arbitrarily small tails and \( f : X \to X \) an expanding continuous map. Then for each \( x_0 \in X \) there exists a fixed point \( \hat{x} = f(\hat{x}) \) of \( f \) with \( \hat{x} \in T x_0 \).

**Proof.** Given \( x_0 \in X \), take a convergent ascending sequence \( x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \) with \( \lim_{n \to \infty} x_n = \hat{x} \in \bigcap_{n \in \mathbb{N}} T x_n \) and \( \text{diam}(T x_n) \leq 1/n \) for each \( n > 0 \). We have \( x_n \leq f(x_n) \) for each \( n \in \mathbb{N} \) and therefore \( f(x_n) \in T x_n \). It follows that the sequence \( \{f(x_n)\}_{n \in \mathbb{N}} \) converges to \( \hat{x} \) and by continuity that \( \hat{x} = f(\hat{x}) \).

We now come to our main concept.

**Definition 1.** We say that \((X; d, \preceq)\) is a partially ordered Cantor space (or simply a Cantor space), provided (i) tails are closed, (ii) \((X; d, \preceq)\) admits arbitrarily small tails and (iii) \( d \) is complete.

The main property of Cantor spaces is given in

**Theorem 1 (Order-theoretic Cantor theorem).** Let \( X = (X; d, \preceq) \) be a Cantor space. Then:

(i) Any tail \( T x \) in \( X \) is also a Cantor space.

(ii) \( X \) contains at least one maximal element.

(iii) Any tail \( T x \) in \( X \) contains at least one maximal element \( x^* \) in \( X \).

(iv) If \( F : X \to X \) is expanding, then each tail \( T x \) contains a fixed point of \( F \).

**Proof.** (i) is obvious from the definitions involved; (iii) and (iv) follow clearly from (i) and (ii). It thus remains to verify that (ii) is true. The existence in \( X \) of a maximal element follows from Proposition 1. Indeed, let \( x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \) be an ascending sequence which converges to a point
\[ \hat{x} \text{ such that } \hat{x} \in \cap_{n \in \mathbb{N}} T x_n. \text{ We claim that } \hat{x} \text{ is maximal in } X: \text{ for, if } z \succeq \hat{x}, \text{ then } z \succeq \hat{x} \succeq x_n \text{ for each } n \geq 0, \text{ so } z \in \cap_{n \in \mathbb{N}} T x_n \text{ and therefore } z = \hat{x}. \] This completes the proof.

2. Bishop–Phelps Theorem

Following Bishop–Phelps, we introduce the following:

**Definition 2.** Let \((X, d)\) be a metric space, \(\varphi : X \to \mathbb{R}\) be a real-valued function and \(\lambda\) a positive number. Following Bishop–Phelps, we define a relation \(\preceq_{\varphi, \lambda}\) on \(X\) by

\[
(x, y) \in \preceq_{\varphi, \lambda} \iff \varphi(y) + \lambda d(x, y) \leq \varphi(x).
\]

This is in fact a partial ordering on \(X\): clearly, \(x \preceq_{\varphi, \lambda} x\) for each \(x \in X\); if \(x \preceq_{\varphi, \lambda} y\) and \(y \preceq_{\varphi, \lambda} x\), then \(2\lambda d(x, y) = \lambda d(x, y) + \lambda d(y, x) \leq \varphi(x) - \varphi(y) - \varphi(x) = 0\) and \(x = y\); finally, if \(x \preceq_{\varphi, \lambda} y\) and \(y \preceq_{\varphi, \lambda} z\), then from the triangle inequality, we find \(x \preceq_{\varphi, \lambda} z\). The space \((X, d)\) together with this partial ordering is denoted by \(X_{\varphi, \lambda}\). In the special case \(\lambda = 1\), we shall write \(\preceq_{\varphi}\) for \(\preceq_{\varphi, \lambda}\) and \(X_{\varphi}\) for \(X_{\varphi, 1}\). Observe that if \(x, y\) are known to be related then the condition \(\varphi(y) \leq \varphi(x)\) alone assures that both \(x \preceq_{\varphi, \lambda} y\) and \(\varphi(y) + \lambda d(x, y) \leq \varphi(x)\).

**Proposition 3.** Let \(\varphi : X \to \mathbb{R}\) be a function and \(\lambda > 0\). Then (i) if \(\varphi : X \to \mathbb{R}\) is bounded below, then \(X_{\varphi, \lambda}\) admits arbitrarily small tails; (ii) if \(\varphi\) is lower semicontinuous, then each tail in \(X_{\varphi, \lambda}\) is closed.

**Proof.** Clearly, for the proof, we may assume that \(\lambda = 1\). (i) Letting \(x \in X\) and \(\varepsilon > 0\) be given, we choose an element \(y \in Tx\) so that

\[
\varphi(y) - \inf_{t \in Tx} \varphi(t) \leq \varepsilon/2.
\]

From \(Ty \subset Tx\) we have, for any \(z_1, z_2 \in Ty\),

\[
d(z_1, z_2) \leq d(z_1, y) + d(z_2, y) \leq 2\varphi(y) - 2 \inf_{t \in Ty} \varphi(t) \leq 2\varphi(y) - 2 \inf_{t \in Tx} \varphi(t) \leq \varepsilon.
\]

From this we get \(\text{diam}(Ty) \leq \varepsilon\) as asserted.

(ii) Indeed, given a tail \(Tx = \{y \mid \varphi(y) + d(x, y) \leq \varphi(x)\}\), because the map \(y \mapsto \varphi(y) + d(x, y)\) is lower semicontinuous, the conclusion follows.

Theorem 1 leads immediately to the following fundamental result:
Theorem 2 (Bishop–Phelps). Let \((X, d)\) be complete, \(\varphi : X \to \mathbb{R}\) a l.s.c. function on \(X\) with a finite lower bound and \(\lambda\) a positive number. Then given any \(x_0\) there exists at least one maximal element \(x^*\) in \(X_{\varphi, \lambda}\) with \(x^* \in Tx_0\). Precisely, for any \(x_0\) there is at least one \(x^* \in X\) such that \[\varphi(x^*) + \lambda d(x_0, x^*) \leq \varphi(x_0)\]

and \[\varphi(x^*) < \varphi(x) + \lambda d(x, x^*)\]

for any \(x \neq x^*\).

**Proof.** Let \(Tx_0 \subset X_{\varphi, \lambda}\) be the tail containing a given element \(x_0\) in \(X\); by Proposition 3, \(Tx_0\) is a Cantor space, and thus our assertion is an immediate consequence of Theorem 1.

3. Applications to Fixed Points

Let us say that a function \(F : X \to X\) defined on a metric space \((X, d)\) fulfils Caristi’s condition with respect to a given function \(\varphi : X \to \mathbb{R}^+\) if

\[d(x, Fx) \leq \varphi(x) - \varphi(Fx)\]

for each \(x \in X\).

If it is clear from the context which function \(\varphi\) is involved, we will simply say that Caristi’s condition holds. Notice that if \((*)\) holds with respect to a function \(\varphi : X \to \mathbb{R}\) which is only assumed to be bounded below, then it obviously holds with respect to \(\varphi - \inf_{x \in X} \varphi(x)\). So, there is no loss of generality if \(\varphi\) is assumed to be positive instead of bounded below. We establish now a version of the Caristi–Brøndsted theorem (cf. [3] and [5]).

**Theorem 3.** Let \((X, d)\) be complete and \(\varphi : X \to \mathbb{R}^+\) be a l.s.c. function on \(X\). Then given any \(x_0\), there exists an \(x^* \in X\) such that \(x_0 \preceq x^*\) and \(x^*\) is a common fixed point for the family of functions (not necessarily continuous) \(F : X \to X\) for which Caristi’s condition holds.

**Proof.** Consider the Cantor space \(X_{\varphi}\) and note that the estimate \((*)\) means that \(F : X_{\varphi} \to X_{\varphi}\) is expanding with respect to the partial order \(\preceq_{\varphi}\). Now, by the introductory remarks on expanding maps, and because a tail \(Tx_0 \subset X_{\varphi}\) is a Cantor space, the conclusion follows.

**Theorem 4 (Brøndsted).** Let \((X, d)\) be complete and \(\varphi : X \to \mathbb{R}^+\) an arbitrary function. Then given any \(x_0 \in X\), there exists an ascending convergent sequence \(x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots\) such that \(\lim_{n \to \infty} x_n\) is a
common fixed point for the family of all continuous functions $F : X \to X$ for which Caristi’s condition holds.

Proof. We have seen that $X_\varphi$ admits arbitrarily small tails and that the estimate (*) means that $F : X_\varphi \to X_\varphi$ is expanding with respect to the partial order $\preceq_\varphi$. Now, the conclusion follows from Propositions 1 and 2. ■

The order-theoretic Cantor theorem is equally useful for dealing with multivalued maps. Following W. Takahashi [15], we give the multivalued extension of Caristi’s theorem and then establish Nadler’s fixed point theorem for set-valued contractions.

**Theorem 5 (W. Takahashi [15]).** Let $(X, d)$ be complete and $\varphi : X \to R$ be a l.s.c. function bounded below on $X$. Let $F : X \to X$ be a multivalued map such that for each $x \in X$ there is $y \in F_x$ satisfying

$$d(x, y) \leq \varphi(x) - \varphi(y).$$

Then given any $x_0$ there exists at least one fixed point $x^*$ of $F$ with $x_0 \preceq_\varphi x^*$.

Proof. For each $x \in X$, choose $F_x \in F_x$ such that $d(x, y) \leq \varphi(x) - \varphi(Fx)$. Obviously, $F$ is a single-valued selector of $F$. By Caristi’s theorem, there is a point $x_0$ such that $x_0 \preceq_\varphi x^*$ and $Fx_0 = x_0$. Obviously, $x_0 \in F x_0$. ■

Given a metric space $(X, d)$, let us denote by $\mathcal{CB}(X)$ the family of closed nonempty bounded subsets of $X$. The Hausdorff metric on $\mathcal{CB}(X)$ is denoted by $d_H$. A map $F : X \to \mathcal{CB}(X)$ is $\alpha$-contractive, where $0 \leq \alpha < 1$, if

$$d_H(F x, F y) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

**Theorem 6 (Nadler [12]).** If $(X, d)$ is a complete metric space, then every $\alpha$-contractive map $F : X \to \mathcal{CB}(X)$ has a fixed point.

Proof. First, notice that for any $x \in X$ and any $y \in F x$, we have $d(y, F y) \leq \alpha d(x, y)$. Indeed, for any $\delta > 0$ we have

$$F x \subset \bigcup_{z \in F y} B(z, \alpha d(x, y) + \delta).$$

Therefore, if $y \in F x$ there is a point $z \in F y$ such that $d(y, z) < \alpha d(x, y) + \delta$. Taking the infimum over $z \in F y$ yields $d(y, F y) \leq \alpha d(x, y) + \delta$. Since $\delta > 0$ was arbitrary, the conclusion follows.
Now, fix $\epsilon > 0$. For any $x \in X$ there exists a point $y_\epsilon(x) \in \mathcal{F}x$ such that
\[ d(x, y_\epsilon(x)) \leq (1 + \epsilon)d(x, \mathcal{F}x). \]
From this we get
\[ \left[ \frac{1}{1 + \epsilon} \right] d(x, y_\epsilon(x)) \leq d(x, \mathcal{F}x) - \alpha d(x, y_\epsilon(x)) \leq d(x, \mathcal{F}x) - d(y_\epsilon(x), \mathcal{F}y_\epsilon(x)). \]
Let
\[ \varphi_\epsilon(x) = \left[ \frac{1}{1 + \epsilon} \right] - \alpha^{-1} d(x, \mathcal{F}x). \]
If $\epsilon$ is chosen such that $(1 + \epsilon)^{-1} > \alpha$, then $\varphi_\epsilon$ is continuous and bounded below. Furthermore, we have just shown that for any $x \in X$,
\[ x \preceq y_\epsilon(x) \quad \text{and} \quad y_\epsilon(x) \in \mathcal{F}x. \]
Let $x^*$ be a maximal element for the partial order $\preceq_{\varphi_\epsilon}$. From $x^* \preceq y_\epsilon(x^*)$ we get $x^* = y_\epsilon(x^*)$, and therefore $x^* \in \mathcal{F}x^*$.

4. Applications to Geometry of Banach Spaces

Let $B = B(z, r)$ be a closed ball in a Banach space. For any $x \notin B$, the convex hull of $x$ and $B$ is called a drop and is denoted by $D(x, B)$; it is clear that if $y \in D(x, B)$, then $D(y, B) \subset D(x, B)$, and, if $z = 0$, that $\|y\| \leq \|x\|$.

**Theorem 7 (Daneš [7]).** Let $A$ be a closed subset of a Banach space $E$, let $z \in E - A$, and let $B = B(z, r)$ be a closed ball of radius $0 < r < d(z, A) = R$. Let $F : A \to A$ be any map such that $F(a) \in A \cap D(a, B)$ for each $a \in A$. Then for each $x \in A$, the map $F$ has at least one fixed point in $A \cap D(x, B)$.

**Proof.** We can assume $z = 0$. Let $\|x\| = \rho \geq R$ and let $X = A \cap D(x, B)$; clearly, $F$ maps $X$ into itself and we shall develop an expression for $\|x - F(x)\|$ on $X$.

Given $y \in X$, there is a $b \in B$ with $F(y) = tb + (1 - t)y$; since $\|F(y)\| \leq t\|b\| + (1 - t)\|y\|$, we have $t[\|y\| - \|b\|] \leq \|y\| - \|F(y)\|$ so because $\|y\| - \|b\| \geq R - r$, we find
\[ t \leq \frac{\|y\| - \|Fy\|}{R - r}. \]
Thus,
\[ \|y - F(y)\| \leq t\|y - b\| \leq t[\|y\| + \|b\|] \leq t[\rho + r] \leq \frac{\rho + r}{R - r}[\|y\| - \|F(y)\|]. \]
Therefore, applying the Theorem of Caristi with 
\[ \varphi(x) = \frac{\rho + r}{R - r} \|x\|, \]
the result follows. \[\blacksquare\]

As a consequence, we obtain

**Theorem 8 (Supporting Drops Theorem).** Let \( A \) be a closed set in a Banach space \( E \), and \( z \in E - A \) a point with \( d(z, A) = R > 0 \). Then for any \( r < R < \rho \) there is an \( x_0 \in \partial A \) with 
\[ \|z - x_0\| \leq \rho \quad \text{and} \quad A \cap D(x_0, B(z, r)) = \{x_0\}. \]

**Proof.** Let \( \tilde{A} = A \cap B(z, \rho) \). It is a closed and nonempty subset of \( E \). For each point \( x \in \tilde{A} \), choose a point \( F(x) \in \tilde{A} \cap D(x, B) \) such that \( F(x) \neq x \) if \( A \cap D(x, B) \neq \{x\} \). One can easily see that a fixed point \( x_0 \) of \( F \) occurs at points of \( \partial A \) and that \( \tilde{A} \cap D(x, B) = A \cap D(x, B) \). \[\blacksquare\]

## 5. Applications to Critical Point Theory

Let \( \varphi : X \to \mathbb{R} \) be a real-valued function\(^2\) on a metric space \( X \) with a finite \( \eta = \inf\{\varphi(x) \mid x \in X\} \). Recall that a minimizer (resp. a strict minimizer) of \( \varphi \) is an element \( x_0 \in X \) with \( \varphi(x_0) = \eta \) (resp. such that the relation \( \varphi(z) \leq \varphi(x_0) \) implies \( z = x_0 \)). A sequence \( \{x_n\} \) in \( X \) for which \( \varphi(x_n) \to \eta \) is called a minimizing sequence for \( \varphi \).

**Theorem 9 (Ekeland [9]).** Let \( (X, d) \) be complete and let \( \varphi : X \to \mathbb{R} \) be a lower semicontinuous function with finite lower bound \( \eta \). Let \( \{x_n\} \) be a minimizing sequence for \( \phi \) and \( \lambda_n = (\varphi(x_n) - \eta)^{1/2} \). Then there exists a minimizing sequence \( \{y_n\} \) for \( \varphi \) such that for any natural \( n \) we have:

(i) \( \varphi(y_n) \leq \varphi(x_n) \) and \( d(x_n, y_n) \leq \lambda_n \),

(ii) \( y_n \) is a strict minimizer of the function \( \varphi_n : X \to \mathbb{R} \) given by 
\[ \varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \quad \text{for} \ z \in X, \]

(iii) \( \varphi(y_n) = \varphi_n(y_n) \leq \varphi(z) + \lambda_n d(z, y_n) \) for \( z \in X \).

\(^2\) For simplicity, we avoid considering the extended real functions \( \varphi : X \to \mathbb{R} \cup \{\infty\} \).
Proof. We first describe the construction of \( \{y_n\} \). For a given natural \( n \), consider the space \( X_{\varphi,\lambda_n} \), where \( \lambda_n = (\varphi(x_n) - \eta)^{1/2} \). By the Bishop–Phelps theorem applied in \( X_{\varphi,\lambda_n} \) for the point \( x_n \), there exists an element \( y_n \) in \( X_{\varphi,\lambda_n} \) such that (a) \( x_n \preceq \varphi,\lambda_n y_n \) and (b) \( y_n \) is maximal in \( X_{\varphi,\lambda_n} \). We now show that \( y_n \) and the function \( \varphi_n \) defined in (ii) have the properties (i)–(iii).

Indeed, the relation \( x_n \preceq \varphi,\lambda_n y_n \) in \( X_{\varphi,\lambda_n} \) translates into the estimate

\[
\lambda_n d(x_n, y_n) \leq \varphi(x_n) - \varphi(y_n),
\]

and gives

\[
d(x_n, y_n) \leq \frac{1}{\lambda_n} (\varphi(x_n) - \varphi(y_n)) \leq \frac{1}{\lambda_n} (\eta + \lambda_n^2 - \eta) = \lambda_n;
\]

thus (i) is satisfied.

To establish (ii), suppose that \( \varphi_n(z) \leq \varphi_n(y_n) \) for some \( z \) in \( X \); we then have

\[
\varphi_n(z) = \varphi(z) + \lambda_n d(z, y_n) \leq \varphi(y_n) = \varphi_n(y_n),
\]

which (by the definition of the order in \( X_{\varphi,\lambda_n} \)) gives \( y_n \preceq \varphi,\lambda_n z \). Since \( y_n \) is maximal in \( X_{\varphi,\lambda_n} \), the last relation implies \( y_n = z \), showing that \( y_n \) is a strict minimizer of \( \varphi_n \), as asserted.

(iii) is an obvious consequence of (ii).

Thus we have constructed a minimizing sequence \( \{y_n\} \) satisfying (i)–(iii).

\[\]
Thus, \( \| D\varphi(y_n) \|_{E^*} \leq \lambda_n \) for each \( n \) and, because \( \lambda_n \to 0 \), our assertion follows.

6. Remarks

(1) The Bishop–Phelps technique presented in Sections 2–5 originated in and evolved from the work of the above authors in the theory of support functionals in Banach spaces. Let \( E \) be a Banach space and \( X \subseteq E \). A point \( x_0 \in X \) is a support point of \( X \) if for some \( f \in E^* \), called a support functional of \( X \), we have \( f(x_0) = \sup\{f(x) \mid x \in X\} \). The following theorem was established by Bishop–Phelps [1]: Let \( C \) be a closed convex subset of \( E \).

(a) the support points of \( C \) are dense in the boundary \( \partial C \) of \( C \), and
(b) the support functionals of \( C \) are norm dense in the set \( \{ f \in E^* \mid \sup_C f < \infty \} \).

In connection with the Bishop–Phelps theorem, we make the following comments:

(i) If \( \text{Int}(C) \neq \emptyset \), then every \( x \in C \) is a support point of \( C \); this follows at once from the Mazur separation theorem.

(ii) If \( C \) is the closed unit ball in \( E \), then the set \( \{ f \in E^* \mid f(x) = \|f\| \text{ for some } x \in \partial C \} \) is norm dense in \( E^* \); this is a special case of the Bishop–Phelps theorem.

(iii) If \( C \) is the closed unit ball in \( E \), then \( \{ \text{each } f \in E^* \text{ is a support functional of } C \} \iff \text{the space } E \text{ is reflexive} \) (theorem of James [11]).

(iv) Let \( \varphi : E \to R \) be convex and lower semicontinuous. Let

\[
\partial \varphi(x) = \{ f \in E^* \mid f(y - x) \leq \varphi(y) - \varphi(x) \text{ for } y \in E \}
\]

be the subdifferential of \( \varphi \) at \( x \in E \). Because the elements of \( \partial \varphi(x) \) can be identified with support functionals of the closed convex epigraph \( \text{epi}(\varphi) \subseteq E \times R \) of \( \varphi \) at \( (x, \varphi(x)) \), the Bishop–Phelps theorem leads to the following theorem: The set \( \{ x \in E \mid \partial \varphi(x) \neq \emptyset \} \) is dense in \( E \). This important result (and, in fact, its “extended” version valid for functions \( \varphi \) possibly equal to \( \infty \)), is due to Brønsted–Rockafellar [4].

(2) The order-theoretic Cantor theorem implies the usual Cantor theorem. Indeed, let \( \{ F_n \}_{n \in N} \) be a decreasing sequence of nonempty closed sets in a complete metric space \( (X, d) \) (we can always assume \( F_0 = X \)) such that \( \inf_{n \in N} \text{diam} F_n = 0 \). Let \( x \preceq y \) if \( x = y \) or there exists \( n \in N \) such that \( y \in F_n \) and \( x \notin F_n \). Then \( \preceq \) is compatible with the metric since \( Tx = \{ x \} \) if \( x \in \cap_{n \in N} F_n \) and \( Tx = \{ x \} \cup F_{n(x)+1} \) otherwise, where \( n(x) = \max\{n \in N \mid x \in F_n\} \). Clearly, any maximal element belongs to \( \cap_{n \in N} F_n \).

(3) The theorem of Daneš can be proved by replacing the norm by a function \( \varphi : E \to R \cup \{\infty\} \) which is l.s.c., coercive, bounded below and convex.
(4) The formulation of the Bishop–Phelps theorem is taken from [8]; the result appeared in a different form in the survey by Phelps [14] written in 1971. For various interrelations between results related to the Bishop–Phelps theorem, the reader is referred to [6] and [13].

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