THE REAL PART OF AN OUTER FUNCTION AND A HELSON-SZEGŐ WEIGHT

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Dedicated to Professor Kôzô Yabuta on the occasion of his sixtieth birthday

Abstract. Suppose $F$ is a nonzero function in the Hardy space $H^1$. We study the set $\{f: f$ is outer and $|F| \cdot \Re f$ a.e. on $\partial D\}$, where $\partial D$ is the unit circle. When $F$ is a strongly outer function in $H^1$ and $\alpha$ is a positive constant, we describe the set $\{f: f$ is outer, $|F| \cdot \alpha \Re f$ and $|F|^{-1} \cdot \alpha \Re (f^{-1})$ a.e. on $\partial D\}$. Suppose $W$ is a Helson-Szegő weight. As an application, we parametrize real-valued functions $v$ in $L^1(\partial D)$ such that the difference between $\log W$ and the harmonic conjugate function $\tilde{v}$ of $v$ belongs to $L^1(\partial D)$ and $\|v\|_1$ is strictly less than $\frac{\alpha}{\pi}$ using a contractive function $\Phi$ in $H^1$ such that $(1 + \Phi) = (1 - \Phi)$ is equal to the Herglotz integral of $W$.

1. Introduction

Let $D$ be the open unit disc in the complex plane and let $\partial D$ be the boundary of $D$. An analytic function $f$ on $D$ is said to be of class $N$ if the integrals

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \log^+ |f(re^{i\mu})| d\mu$$

are bounded for $r < 1$. If $f$ is in $N$, then $f(e^{i\mu})$, which we define to be $\lim_{r \uparrow 1} f(re^{i\mu})$, exists almost everywhere on $\partial D$. If

$$\lim_{r \uparrow 1} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \log^+ |f(re^{i\mu})| d\mu = \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \log^+ |f(e^{i\mu})| d\mu$$

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then \( f \) is said to be of class \( N_+ \). The set of all boundary functions in \( N \) or \( N_+ \) is denoted by \( N \) or \( N_+ \), respectively. For \( 0 < \rho < 1 \), the Hardy space \( H^\rho \) is defined by \( N_+ \setminus L^\rho \). Hence any function in \( H^\rho \) has an analytic extension to \( D \).

A function \( h \) in \( N_+ \) is called \emph{outer} if \( h \) is invertible in \( N_+ \). A function \( g \) in \( H^1 \) is called \emph{strongly outer} if the only functions \( f \in H^1 \) such that \( f \equiv g \) a.e. on \( \mathcal{D} \) are scalar multiples of \( g \). If \( g \) is strongly outer then it is outer. Suppose \( F \) is a nonzero function in \( H^1 \). Define \( \mathcal{D} \) by

\[
\frac{1 + \mathcal{D}(z)}{1 - \mathcal{D}(z)} = \frac{1}{2} \frac{Z}{2} \mathcal{E}^{i\mu + z} \mathcal{F}(e^{i\mu}) \mathcal{J} \mathcal{D} (z \in D);
\]

The right-hand side is the Herglotz integral of \( \mathcal{F} \). Then \( \mathcal{D} \) is a contractive function in \( H^1 \). Let \( f_0 = (1 + \mathcal{D})^{-1} \mathcal{J} \). Then \( \Re f_0(z) > 0 \) for \( z \in D \),

\[
\mathcal{F} = \Re f_0 = \frac{1}{2} \frac{Z}{2} \mathcal{E}^{i\mu + z} \mathcal{F}(e^{i\mu}) \mathcal{J} \mathcal{D} (z \in D);
\]

and \( f_0 \in H^\rho \) by a theorem of Kolmogorov (c.f. [1, Theorem 4.2]). Since \( \Re f_0(z) > 0 \), \( f_0 = ce^{iv} \), where \( c \) is a positive constant, \( kv_1 < \frac{1}{2} \) and \( v \) is a harmonic conjugate function of \( v \) satisfying \( v(0) = 0 \). By a theorem of Kolmogorov, \( v i v 2p \in H^p \),

\[
\mathcal{F} = e^{i+v} \quad \text{and} \quad e^{i} = c \cos v \quad \text{a.e. on} \quad \mathcal{D};
\]

where \( u \) is a real-valued function. In Section 2, when \( F \) is strongly outer we study an outer function \( f \) in \( N_+ \) such that \( \mathcal{F} \cdot \Re F \) a.e. on \( \mathcal{D} \). We then show that \( \mathcal{F} = \Re F \) if and only if \( \mathcal{D} \) is an \( \mathcal{O} \)-Stolz function, where \( \mathcal{O} \) is a positive constant. If \( \mathcal{D} \) is a contractive function in \( H^1 \) and \( \mathcal{J} \in \mathcal{J}(1) \) a.e. on \( \mathcal{D} \), then we call \( \mathcal{D} \) a \( \mathcal{O} \)-Stolz function. Suppose \( \mathcal{W} \) is a Helson-Szeg"{o} weight (c.f. [3]). In Section 3, using Theorem 1 in Section 2, we parametrize real-valued functions \( v \) such that \( \log \mathcal{W} 1 \) \( \forall 2 \in L^1 \) and \( kv_1 < \frac{1}{2} \).

2. The Real Part of an Outer Function

In this section, we study the inequality \( \mathcal{F} \cdot \Re F \) a.e. on \( \mathcal{D} \) when \( F \) is a nonzero function in \( H^1 \). The first author [4] studied the inequality \( \mathcal{F} \cdot \Re f \) a.e. on \( \mathcal{D} \) when \( F \) is strongly outer and \( f \) is outer in \( N_+ \). We give necessary and sufficient conditions of this inequality. We study two inequalities \( \mathcal{F} \cdot \Re f \) and \( \mathcal{F} \cdot \Re f \) a.e. on \( \mathcal{D} \) when \( F \) is strongly outer and \( f \) is in \( N_+ \). Results in this section will be used in the later section.

**Proposition 1.** Suppose \( F \) is a nonzero function in \( H^1 \) and \( \mathcal{O} \) is a constant satisfying \( \mathcal{O} < 1 \). Then the following \( (1) \Rightarrow (3) \) are equivalent:
(1) \( jFj \cdot \Re F \text{ a.e. on } \mathfrak{D} \).

(2) \( F = (1 + \mathfrak{D})^i \mathfrak{D} \) a.e. on \( \mathfrak{D} \) for a contractive function \( \mathfrak{D} \) in \( H^1 \) such that \( \mathfrak{D} \) is a \( 0^- \)-Stolz function.

(3) \( F = c e^{i v} \text{ a.e. on } \mathfrak{D} ; \) where \( c \) is a positive constant and \( v \) is a real function in \( L^1 \) satisfying \( k v k_1 \cdot \cos \frac{1}{2}(1 - \theta) \) \( < \frac{1}{2} \).

*Proof.* (1), (2): Since \( F \in H^1 \) and \( \Re F \), 0 a.e. on \( \mathfrak{D} \), it follows that

\[
\Re F(z) = \frac{1}{2} \int_0^1 \frac{Z_{2v} 1 \cdot jz^2}{jz_1 \cdot jz} Re F(e^{i\mu}) \, d\mu, \quad 0 \leq \Re F(z) \leq 1 \text{ a.e. on } \mathfrak{D} .
\]

Hence \( F = (1 + \mathfrak{D})^i \mathfrak{D} \) for a contractive function \( \mathfrak{D} \) in \( H^1 \). Since \( jFj \cdot \Re F \) a.e. on \( \mathfrak{D} \),

\[
\frac{-1 + \mathfrak{D}^2}{1 \cdot \mathfrak{D}}, \quad Re \frac{1 + \mathfrak{D}^2}{1 \cdot \mathfrak{D}} = \frac{1}{1 \cdot \mathfrak{D}} \cdot jFj \cdot \Re F \text{ a.e. on } \mathfrak{D} .
\]

Hence \( jFj \cdot \Re F \text{ a.e. on } \mathfrak{D} \) and so \( \mathfrak{D} \) is a \( \cdot \)-Stolz function. The converse is clear.

(2) (3): Since \( kv k_1 \cdot 1, Re F = \frac{1}{1 + \mathfrak{D}^2} \) a.e. on \( \mathfrak{D} \). Since \( F \in H^1 \), this implies that \( \Re F(z) \) 0, \( 0 \leq \Re F(z) \leq 1 \text{ a.e. on } \mathfrak{D} . \) Since \( \mathfrak{D} \) is a \( \cdot \)-Stolz function, it follows that

\[
jFj = \frac{-1 + \mathfrak{D}^2}{1 \cdot \mathfrak{D}}, \quad jFj \cdot \Re F \text{ a.e. on } \mathfrak{D} .
\]

Hence \( 1 \cdot \cos v \text{ a.e. on } \mathfrak{D} , \) this implies that \( kv k_1 \cdot \cos \frac{1}{2}(1 - \theta) \) \( < \frac{1}{2} \).

(3) (1): By (3), \( jFj = c e^{iv} \cdot Re F = c e^{iv} \cdot \cos v = \Re F \). This implies (1). \( \Box \)

By Proposition 1 (3) and [2, Corollary III. 2.6], if \( jFj \cdot \Re F \) a.e. on \( \mathfrak{D} \) then both \( F \) and \( F^i \) belong to \( H^P \) for some \( p > 1 \).

**Proposition 2.** Suppose \( F \) is a strongly outer function in \( H^1 \). Define \( \mathfrak{D} \) by

\[
\frac{1 + \mathfrak{D}(z)}{1 \cdot \mathfrak{D}(z)} = \frac{1}{2} \int_0^1 Z_{2v} e^{i\mu} + z_{e^{i\mu}} jF(e^{i\mu}) \, d\mu, \quad 0 \leq \Re F \leq 1 \text{ a.e. on } \mathfrak{D} .
\]

For \( f \) in \( N^+ \), (1) \( \Rightarrow \) (3) are equivalent:

(1) \( jFj \cdot Re f \) a.e. on \( \mathfrak{D} \) and \( f \) is an outer function.

(2) \( f = [(1 + \mathfrak{D}) \equiv \mathfrak{D}] + [(1 + \mathfrak{D})] \text{ a.e. on } \mathfrak{D} \) for some contractive function \( \mathfrak{D} \) in \( H^1 \).

(3) \( jFj \equiv e^{i\mu} \cdot \mathfrak{D}, \quad jFj \equiv e^{i\mu}, \quad c \cos v \) and \( f = c e^{iv} \) a.e. on \( \mathfrak{D} \) where \( c \) is a positive constant and \( \mu \) and \( v \) are real functions.
The following proof is similar to the one of Theorem 6 in the first author’s paper [4].

Proof. (1) (3): Let Arg \( f \) denote the argument of \( f \) restricted to \( j \frac{1}{4} < \arg f < \frac{1}{4} \). Let \( \nu = j \arg f \). Then \( j\nu \cdot \frac{1}{4} \) and \( f = jf je^{j\nu} \). Since \( 0 < jFj \cdot \Re f, j\nu < \frac{1}{4} \). By the proof of [2, Lemma IV. 5.4], if \( j\nu \cdot \frac{1}{4} \) then \( e^\nu \cos \nu 2 L^1 \). Let \( g = e^{j\nu} \nu \). Then \( f g = jf je^{j\nu} > 0 \). Since \( f \) is outer, \( F=fg 2 N_+ \). Since

\[
\frac{-F}{fg} \cdot \Re \frac{f}{g} = \frac{\cos \nu}{j\nu} = e^\nu \cos \nu 2 L^1;
\]

it follows that \( F=fg 2 H^1 \). Since \( F \) is strongly outer, \( F=fg \) is a scalar multiple of \( F \). Hence \( f = c \) for some positive constant \( c \). Hence \( f = c e^{j\nu} \), and hence \( jFj \cdot c \cos \nu \). Define \( u \) by \( jFj = e^{j\nu} \). Then \( e^{j\nu} \cdot c \cos \nu \). This implies (3).

(3) (2): In the following we do not assume that \( F \) is strongly outer. We assume that \( F \) is a nonzero function in \( H^1 \). By (3), \( jFj \cdot \Re f \) and \( \Re f 2 L^1 \). Let \( (\nu_i j\nu_i)z \) denote the Poisson transform of \( \nu_i j\nu_i(e^{j\nu_i}) \). Let \( g(z) = c e^{j\nu_i} \). Then \( \Re g(z) = 0 (z 2 D), \lim \sup \frac{1}{r} \int_1^Z \frac{2^{j\nu_i} 1_i jzj^2}{e^{j\nu_i} iz^2} g(z) d\mu = \Re g(0) < 1 : \)

Hence

\[
\Re g(z) = \frac{1}{2^{j\nu_i} 0} \int \frac{2^{j\nu_i} 1_i jzj^2}{e^{j\nu_i} iz^2} \frac{\Re f(e^{j\nu_i})} {jF(e^{j\nu_i})} j\nu_i \frac{\mu 1_{+} \phi(z)} {1_i \phi(z)} (z 2 D);
\]

Hence there exists a contractive function \( - \) in \( H^1 \) such that

\[
g(z) = \frac{1 + \phi(z)} {1_i \phi(z)} + \frac{1 + \phi(z)} {1_i \phi(z)} (z 2 D);
\]

Since \( \lim \int \frac{g(r\phi)} {1_i \phi(z)} = f(e^{j\nu_i}) \) a.e. on \( @D \), this implies (2).

(2) (1): Since \( jFj \cdot 1, \Re (1 + \phi) = 1_i \phi \), 0. Hence

\[
jFj = \Re \frac{1 + \phi} {1_i \phi} \cdot \Re \frac{1 + \phi} {1_i \phi} = \Re f \text{ a.e. on } @D;
\]

This implies (1).
By Proposition 2 (3) and [2, Corollary III. 2.6], if \( j \mathbf{F} j \cdot \Re f \) a.e. on \( \mathcal{D} \) and \( f \) is an outer function then both \( f \) and \( f^{1/2} \) belong to \( \mathcal{H}^p \) for all \( p < 1 \).

By (1), the set of all functions \( f \) satisfying one of the conditions (1) \( \Rightarrow \) (3) is a convex subset of \( \mathcal{N}_+ \). If \( \mathcal{F} \) is a nonzero function in \( \mathcal{H}^1 \), then (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1) holds in Proposition 2. But by [4, Theorem 6], (1) \( \Rightarrow \) (3) does not hold in general.

**Theorem 1.** Suppose \( \mathcal{F} \) is a strongly outer function in \( \mathcal{H}^1 \). Define \( \mathcal{D} \) by

\[
\frac{1 + \mathcal{D}(z)}{1 + \mathcal{D}(z)} = \frac{1}{2i} \int_0^{2\pi} e^{i\mu} \frac{z}{e^{\mu^2} z^2} \mathcal{F}(e^{\mu})j d\mu \quad (z \neq 2D);
\]

For \( f \) in \( \mathcal{N}_+ ; \) (1) \( \Rightarrow \) (4) are equivalent. \( (^o_1; \ldots; ^o_5 \) are positive appropriate constants.)

(1) \( \Re f \) a.e. on \( \mathcal{D} \).

(2) (1-2) \Re \mathbf{F} j \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re \mathbf{F} i^{1/2} \Re (\mathbf{F} i^{1/2}) \) a.e. on \( \mathcal{D} \).

(3) There exists a contractive function \( \mathcal{D} \) in \( \mathcal{H}^1 \) such that

\[
^o_3 \mathbf{F} = \frac{1 + \mathcal{D}}{1 + \mathcal{D}(1)} \quad \text{and} \quad \frac{1 + \mathcal{D}}{1 + \mathcal{D}(1)} \quad \text{a.e. on } \mathcal{D};
\]

(4) There exists a constant \( C > 0 \) and real functions \( u, v \) in \( \mathcal{L}^1 \) such that

\[
\mathbf{F} j = e^{i\mu^2}; \quad \kappa v k_1 + c^1 \cos \frac{1}{2} \mathcal{D} < \frac{1}{2} \quad \text{and} \quad f = c e^{i\mu} \quad \text{a.e. on } \mathcal{D}.
\]

**Proof.** (1) \( \Rightarrow \) (2): By (1),

\[
(\Re f)^2 \cdot j \mathbf{F} j \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} (\Re f) \mathbf{F} j \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} (\Re f)^2:
\]

Hence \( \mathbf{F} j \cdot \Re f \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} (\Re f) \mathbf{F} j \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} (\Re f)^2 \).

(2) \( \Rightarrow \) (1): By (2),

\[
\frac{1}{\mathbf{F} j} \cdot \Re f \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re f = \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re f;
\]

This implies (1) with \( ^o_1 = ^o_3 \).

(2) \( \Rightarrow \) (3): Since \( f \) \( \mathcal{H}^1 \) and \( \Re f \) \( 0 \) a.e. on \( \mathcal{D} \), \( \Re f(z) > 0 \) (2 \( \mathcal{D} \)). Hence \( f \) is an outer function. Since \( \mathbf{F} j \cdot \Re f \), by Proposition 2,

\[
\frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re f = \frac{2(1 + \mathcal{D})}{1 + \mathcal{D}(1)};
\]

for some contractive function \( \mathcal{D} \) in \( \mathcal{H}^1 \). Since \( \mathbf{F} j \cdot \Re f \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \mathbf{F} j \),

\[
\frac{2j_1 \mathcal{D} j}{j_1 \mathcal{D} j} = \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re f \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \mathbf{F} j = \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \Re f \cdot \frac{1 + \mathcal{D}}{1 + \mathcal{D}} \mathbf{F} j.
\]
This implies (3) with $\theta_3 = \theta_2 = 2$ and $\theta_4 = \theta_3 + 2$.

(3) By (3), $f$ is outer, since $\mathcal{R}$ and $\mathcal{L}$ are contractive. Since

$$jF_j = \text{Re} \left( \frac{\mu_1 + \mathcal{R}}{1_1 \mathcal{L}} \right) \cdot 2\theta_3 \text{Re} f,$$

by Proposition 2, $jF_j = e^{i+\nu} \cdot jv < 1 \neq 2$, $c_0 \cos \nu$ and $2\theta_3 f = c_0 e^{i+\nu}$, where $c_0$ is a positive constant and $\nu$ are real functions. Hence

$$c_0 e^{i+\nu} = 2\theta_3 jF_j = 2\theta_3 e^{i+\nu}.$$

Hence $c_0 = 2\theta_3 \cdot e^{i+\nu}$. By (3) and $c_0 \cos \nu$, $1 = 2\theta_4 > 0$. Hence $\nu, \nu$ is a Helson-Szegö weight. Then for each $\nu$ in $E_W$ there exists $u \in \mathbb{R}$ such that $\log W = u + \nu$. In this section, we study two problems about a Helson-Szegö weight. In Theorem 2 we describe $E_W$. Theorem 3 follows from Theorem 2 immediately.
Theorem 2. Let $W$ be a positive function in $L^1$. Define $\oplus$ by
\[
1 + \oplus(z) = \frac{1}{2\pi i} \int_{0}^{2\pi} e^{i\mu} + z e^{i\mu} W(e^{i\mu}) d\mu \quad (z \in D).
\]
Then $v$ belongs to $E_W$ if and only if
\[
v = i \text{Arg} \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{a.e. on } \partial D;
\]
where $\bar{\cdot}$ is a contractive function in $H^1$ satisfying
\[
\frac{|1 - \bar{\cdot}|}{|1 - \cdot|} \leq 1 \quad \text{a.e. on } \partial D,
\]
for some constant $\circ > 0$.

Proof. If $v \in E_W$, then $v \in E_{W^\circ}$ for some constant $\circ > 0$. Hence $W = e^{v+i\psi}$ where $u \in L^1$ and $k\psi_k \cdot (\beta \bar{\psi})_i$ "$. Hence there exists a constant $\circ > 0$ such that
\[
W \cdot \circ e^{v\cos v} \quad \text{and} \quad W^1 \cdot \circ e^{v\cos v};
\]
where $\psi_k \cdot \circ \cos v$. If $f = e^{v+i\psi}$, then $W \cdot \circ \text{Re } f$, $W^1 \cdot \circ \text{Re } f$ and $f \in H^1$. Since $W$, $W^1 \in L^1$, there exists an outer function $F$ such that $f F = W$ and $F F^1 \in H^1$. Hence $F$ is strongly outer. By Theorem 1, there exist constants $\circ_3$, $\circ_4 > 0$ and a contractive function $\bar{\cdot} \in H^1$ such that
\[
\circ_3 f = \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{and} \quad \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{a.e. on } \partial D;
\]
Hence
\[
v = i \text{Arg } f = i \text{Arg} \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{a.e. on } \partial D;
\]
This implies the "only if" part. Conversely, suppose $v$ satisfies the condition. Define $f$ by
\[
f = \frac{1}{1 + \oplus(1 - \oplus)};
\]
Then
\[
v = i \text{Arg } f \quad \text{and} \quad j f j \cdot \circ \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{a.e. on } \partial D
\]
for some constant $\circ > 0$. Then $f$ satisfies (3) of Theorem 1 and
\[
W = \frac{1}{1 + \oplus(1 - \oplus)} \quad \text{and} \quad 2 \text{Re } f \cdot j f j \cdot \circ \frac{1}{1 + \oplus(1 - \oplus)} = \circ W;
\]
Since $W$ is a positive function in $L^1$, Re $f$, 0 a.e. on @ and $f \in H^1$. Hence $f$ is strongly outer. Since $\log W \in L^1$, there exists an outer function $F \in H^1$ such that $jFj = W$. Let $k$ be any function satisfying $k \in H^1$ and $k = F$, 0 a.e. on @. Since $f \notin H^1$, $kf \notin H^1$. Since $f$ is strongly outer, $kf = cf$ for some constant $c$. Hence $k = cf$. Therefore $F$ is strongly outer. By Theorem 1, there exists a constant $c > 0$ and real functions $u, v_0 \in L^1$ such that $kv_0k_1 < \sqrt{2}$, $W = e^{i+v_0}$ and $f = c e^{i+iv_0}$ a.e. on @. Hence

$\nu_0 = i \text{ Arg } f = i \text{ Arg } \frac{1}{(1 - \text{ e})(1 - i)} = \nu$.

Hence $W = e^{i+\nu}$ a.e. on @ and $kv_0k_1 < \sqrt{2}$. Hence $\nu$ belongs to $E_W$. 

By Theorem 2, if $W = 1$ then @ = 0 and hence

$$E_1 = \nu 2 \text{ Re } L^1; \quad kv_0k_1 < \frac{\sqrt{2}}{2} \quad \text{and} \quad \nu 2 L^1 0$$

$$= i \text{ Arg } \frac{1}{1 - \text{ e}}; \quad 2 H^1; \quad k'k \cdot 1 \quad \text{and} \quad \frac{1}{1 - \text{ e}} 2 L^1$$

Theorem 3. Let $W$ be a positive function in $L^1$. Define @ by

$$\frac{1 + @z}{1 - @z} = \frac{Z}{2 + i \mu} e^{\mu} + \frac{Z}{2 - i \mu} e^{-\mu} \frac{W(e^{\mu})}{W(z)} (z \in D):$$

(1) $W$ is a Helson-Szegő weight; that is, $E_W \in D$; if and only if there exists a constant $\sigma > 0$ and a contractive function $\mu$ in $H^1$ such that

$$\frac{j1_{1 - \text{ e}}}{j1 - \text{ e}} \cdot \frac{j1_{1 - \text{ e}}}{j1 - \text{ e}} \cdot \frac{j1_{1 - \text{ e}}}{j1 - \text{ e}} a.e. \text{ on @}$$

(2) If @ is a Stolz function; then $W$ is a Helson-Szegő weight, and $W^1$ belongs to $L^1$.

Proof. By Theorem 2, (1) follows immediately. By Theorem 2 with $\sigma = 0$, if @ is a Stolz function, then $\nu = i \text{ Arg } (1 - \text{ e})$ belongs to $E_W$, and hence $E_W \in D$. By (1), $W$ is a Helson-Szegő weight. Since $W = (1 - j \text{ e})^2 = 1 - j \text{ e}$, $j \text{ e} = [(1 - j \text{ e}) = 1 - j \text{ e}] (1 - j \text{ e}) = 1 - j \text{ e}$ a.e. on @ and @ is a Stolz function, it follows that $W^1 2 L^1$.

Note that if @ is a Stolz function, then @ is also a Stolz function. In fact, if @ is a $\sigma$-Stolz function, then $j \text{ e} \cdot 1$ and $j1_{1 - \text{ e}} \cdot j1_{1 - \text{ e}} + j \text{ e} j1_{1 - \text{ e}} \cdot j1_{1 - \text{ e}} \cdot 2^\sigma (1 - j \text{ e}) \cdot 2^\sigma (1 - j \text{ e})^2$.
Let $W$ be a positive function in $L^1$. By Proposition 1, $W = c e^v$ for a constant $c > 0$ and a real function $v$ with $kv_{k_1} < \frac{1}{\sqrt{2}}$ if and only if there exists an $\mathcal{R}^2$ in $H^1$ such that $\mathcal{R}^2$ is a Stolz function and $W = j1 + \mathcal{R}j1_i$. Then there exists a $u_2 \in \text{Re } L^1$ such that

$$W = \frac{j1_i \mathcal{R}j1_i j \mathcal{R}^2}{j1_i \mathcal{R}j1_i} = e^{u_1 j1_i \mathcal{R}^2} = e^{u_i \text{Re } F};$$

where $F = \frac{1 + \mathcal{R}}{1_i \mathcal{R}}$

4. Remark

Put $B_r = f^- 2 H^1; k^- k_1 \cdot rg$ and put

$$B_\mathcal{R}^2 = -2 B_1; \frac{j1_i \mathcal{R}j1_i}{j1_i \mathcal{R}j1_i - j} \cdot 1_i j \mathcal{R}^2 \quad \text{a.e. on } \partial \text{ for some constant } \sigma > 0;$$

where $\mathcal{R}$ is a contractive function in $H^1$. The set $B_\mathcal{R}^2$ was important in Theorems 1, 2 and 3. Let $W$ be a Helson-Szegő weight. Define $\mathcal{R}$ by

$$1 + \mathcal{R}(z) = \frac{1}{2} \int_0^{3/4} \frac{Z_z \mathcal{R}^2 d\mu + Z_z}{Z_z} \mathcal{R}(z) d\mu$$

Then by Theorem 2,

$$E_W = v = i \text{ Arg} \frac{1_i \mathcal{R}}{(1_i \mathcal{R})(1 - j)}; -2 B_\mathcal{R}^2.$$

If $W = 1$ then $\mathcal{R} = 0$ and

$$E_1 = i \text{ Arg} \frac{1}{1_i} = -2 B^0.$$

In this section, we study such a set $B_\mathcal{R}^2$. $\mathcal{R}$ is a Stolz function if and only if $0 \leq 2 B_\mathcal{R}^2$. $\mathcal{R}$ is a Stolz function if and only if $\mathcal{R} \geq 2 B_\mathcal{R}^2$. Hence if $0 < 2 B_\mathcal{R}^2$ then $\mathcal{R} \geq 2 B_\mathcal{R}^2$. If $\mathcal{R}$ is a Stolz function and $2 B_r, r < 1$, then for some constant $\sigma > 0$

$$\frac{j1_i \mathcal{R}j1_i}{j1_i \mathcal{R}j1_i - j} \cdot \frac{2}{(1_i \mathcal{R}j1_i) \mathcal{R}} \cdot \frac{2^\sigma}{(1_i \mathcal{R}j1_i) \mathcal{R}} \text{ a.e. on } \partial;$$

and hence $2 B_\mathcal{R}^2$. Hence if $\mathcal{R}$ is a Stolz function, then $B_r \mathcal{R} \geq 2 B_\mathcal{R}^2 (r < 1)$.

For two positive functions $f$ and $g$ on $\partial$, if there exists a constant $\sigma > 0$ such that $(1^\sigma)g \cdot f \cdot \sigma g$ a.e. on $\partial$, then we write $f \succ g$.

Lemma. Suppose $\mathcal{R}$ and $-2$ are contractive functions in $H^1$. Then the following

$(1) \succ (5)$ are equivalent:
(1) \( k(\oplus_i^4) \leq (1_i \oplus^5) k_1 < 1 \).

(2) \( j1_i \oplus^2 \cdot \circ_2 (1_i \oplus^2)(1_i \oplus^2) \) \( a.e. \) on \( @ \) for some constant \( \circ_2 > 0 \).

(3) There exists a constant \( \circ_3 > 0 \) such that for any function \( t > 0 \)

\[
\frac{j1_i \oplus j}{j1_i \oplus j} \cdot \circ_3 t 1_i j \oplus^2 + 1_i 1_i \oplus j^2 \frac{3}{2} \leq \frac{j1_i \oplus j}{j1_i \oplus j} \cdot \circ_3 t 1_i j \oplus^2 + 1_i 1_i \oplus j^2 \frac{3}{2} \quad a.e. \) on \( @ \):
\]

(4) There exists a constant \( \circ_4 > 0 \) such that

\[
\frac{j1_i \oplus j}{j1_i \oplus j} \cdot \circ_4 1_i j \oplus^2 \quad a.e. \) on \( @ \)
\]

and

\[
\frac{j1_i \oplus j}{j1_i \oplus j} \cdot \circ_4 1_i j \oplus^2 \quad a.e. \) on \( @ \):
\]

(5) \( j1_i \oplus f = j1_i \oplus \) and \( \circ_5 j1_i \oplus j \leq 1_i j \oplus j = j1_i \oplus j \).

Proof. (1) and (2) are equivalent because

\[
1_i \oplus_4 = \frac{(1_i j \oplus^2)(1_i j \oplus^2)}{1_i j \oplus^2}.
\]

(cf. [5, p. 58]). (2) and (3) are equivalent because if \( a; b > 0 \) then \( 2^p \bar{a} \bar{b} \cdot a + b \) and the equality holds when \( a = b \) (1) \( ) \) (5): Let \( f = (\oplus_i^4) = (1_i \oplus^5) \). Then

\[
k_l k_1 < 1, \quad = (\oplus_i^4 f) = (1_i \oplus^5 f) \) and
\]

\[
j1_i \oplus j = j1_i \oplus f(1_i \oplus j)\frac{1_i j \oplus^2}{1_i j \oplus^2} = j1_i \oplus j f j \oplus j = \frac{1_i j \oplus^2}{2} \cdot \frac{1_i j \oplus^2}{2} = j1_i \oplus j;
\]

Let \( g = (\oplus_i^4) = (1_i \oplus^5) \). Then \( k_l k_1 = k_l k_1 < 1, \oplus = (g + ^4) \oplus 1 + g \) and

\[
j1_i \oplus j = j1_i \oplus j (1_i j \oplus^2) g (1_i j \oplus^2) j1_i \oplus j \frac{1_i j \oplus^2}{2} = j1_i \oplus j 1_i j \oplus^2 \frac{1_i j \oplus^2}{2} = j1_i \oplus j; \]

Hence \( j1_i \oplus f = j1_i \oplus j \). Since \( 0 < 1_i j \oplus f k_1 \cdot j1_i \oplus j \cdot 2 \) and

\[
1_i j \oplus^2 = \frac{(1_i j \oplus^2)(1_i j \oplus^2)}{j1_i \oplus^2}.
\]

(1) \( j1_i \oplus j = 1 \oplus j \) and \( 1_i j \oplus^2 = 1_i j \oplus^2 \). It is clear that (5) implies (4). If we multiply both sides of the two inequalities in (4), then (2) follows.

By the above lemma, Proposition 3 follows immediately.
Proposition 3. If $\circ \ominus 2 \circ B_1 \ominus \frac{1}{2}$, then $B \circ \hspace{0.5cm} \frac{1}{4} \circ 2 \circ B_1 \circ \frac{1}{4} \ominus 1 \circ \frac{1}{i} < 1$.

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