EXACT PROFILE VALUES OF SOME GRAPH COMPOSITIONS

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Abstract. It is known that the determination of the profile for arbitrary graphs is NP-complete. The composition of two graphs $G$ and $H$ is the graph with vertex set $V(G) \cup V(H)$ and $(u_1, v_1)$ is adjacent to $(u_2, v_2)$ if either $u_1$ is adjacent to $u_2$ in $G$ or $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$. The exact values of the profile of the composition of a path with other graphs, a cycle with other graphs, a complete graph with other graphs and a complete bipartite graph with other graphs are established.

1. INTRODUCTION AND TERMINOLOGY

For a graph $G$, $V(G)$ denotes the set of vertices of $G$ and $E(G)$ denotes the set of edges of $G$.

Let $G = (V, E)$ be a graph on $n$ vertices. A one-to-one onto mapping $f : V \rightarrow \{1, 2, \ldots, n\}$ is called a proper numbering of $G$. For a proper numbering $f$, the profile width $w_f(v)$ of a vertex $v$ in a graph $G$ is the number

$$w_f(v) = \max_{x \in N[v]} (f(v) - f(x));$$

where $N[v] = \{x \in V : x = v \text{ or } xv \in E\}$ is the closed neighborhood of $v$. Since $v \in N[v]$, we have $w_f(v) = 0$ if $f(v) = f(x)$ for all $x \in N[v]$. The profile $P_f(G)$ of a proper numbering $f$ of $G$ is defined by

$$P_f(G) = \min_{v \in V} w_f(v);$$

and the profile $P(G)$ of $G$ is the number

$$P(G) = \inf_{f : \text{is a proper numbering of } G} P_f(G).$$

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A proper numbering $f$ is called a profile numbering of $G$ if $P_f(G) = P(G)$. See Figure 1 for some examples.

Lin and Yuan [3] have shown that the profile minimization problem of an arbitrary graph is equivalent to the interval graph completion problem, which was shown to be NP-complete by Garey and Johnson [1]. Since minimizing the profile of a graph has some important applications, a large number of approximation algorithms have been developed, published and used. But the exact value of profile is known for only a few classes of graphs. See Lai and Williams [2].

**Definition 1.** The composition $G[H]$ of a graph $G$ with a graph $H$ is the graph with vertex set $V(G) \cup V(H)$ such that $(u_1; v_1)$ is adjacent to $(u_2; v_2)$ if either $u_1$ is adjacent to $u_2$ in $G$ or if $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$.

Figure 2 shows $P_3[P_4]$. For a composition graph $G[H] = (V; E)$, with graph $G$ of order $m$ and $H$ of order $n$, we represent the vertex set as $V = f_{i,j} : 1 \leq i \leq m; 1 \leq j \leq n$, where column $j$ is denoted by $Q_j (1 \leq j \leq n)$, which represents a copy of $G$, and row $i (1 \leq i \leq m)$ is denoted by $R_i$, which represents a copy of $H$.

The following result from [4] is used extensively for the work done in this paper.

**FIG. 1.** Profile numberings for $P_4$, $C_5$, $K_{1,4}$ and $K_{2,3}$.

**FIG. 2.** $P_3[P_4]$
Proposition 1 (Lin and Yuan [4]). Let $G$ be a graph of order $n$. For any proper numbering $f$ of $G$:

$$P_f(G) = \sum_{i=1}^{n} jN(S_i)j; \text{ where } S_i = f v : v \in \mathcal{V}(G) \text{ and } f(v) \cdot t;$$

2. PATHS WITH OTHER GRAPHS

In this section, we establish the profile of the composition of paths with other graphs.

Theorem 1. Let $G = P_m[H]$, where $H$ is a graph with $n$ vertices. Then

$$P(G) =
\begin{cases}
  m - 1 & \text{for } n = 1; \\
  P(H) & \text{for } m = 1; \\
  P(H) + \frac{3n^2}{2} & \text{for } m = 2 \text{ and } n > 1; \\
  2P(H) + \frac{mn(3n - 1)}{2} & \text{for } m, 3 \text{ and } n > 1.
\end{cases}$$

Proof. For $n = 1$, $G = P_m$; so $P(G) = m - 1$. For $m = 1$, $G = P_1[H] = H$; so $P(G) = P(H)$. For $m = 2$, we note that every vertex in $R_2$ is adjacent to every vertex in $R_1$ so once we number a vertex in both rows, all of the unnumbered vertices in the rows are in $N_f(S)$. By Proposition 1, we know that the profile numbering of $G$ must completely number one row before numbering any vertex in the other row. This will minimize $P_{i=1}^{m-1} jN(S_i)j$, which in turn minimizes $P_f(G)$. Without loss of generality, assume that we first completely number $R_1$ and then $R_2$. We want to number the vertices in $R_1$ in the order of a profile numbering (say, $f$) of $H$. Since every vertex in $R_2$ is adjacent to the vertex $f^{-1}(1)$, it does not matter how we number the vertices in $R_2$. Hence, $P(G) = P(H) + \sum_{i=1}^{n-1} (n + i) = P(H) + (3n^2 - n) = 2$.

For $m = 3$, we first show that $P(G) \cdot P(H) + mn(3n - 1) = 2n^2 + n$. Assume that $g$ is a profile numbering of $H$. Consider a numbering $f$ such that

$$f(v_{i,j}) =
\begin{cases}
g(v_j) + (i - 1)n & \text{for } 1 \leq i \leq m; 1 \leq j \leq n; \\
g(v_j) + (m - 1)n & \text{for } i = m; 1 \leq j \leq n;
\end{cases}$$

Then

$$P_f(G) = P(H) + (m - 1)(n + i) + P(H) + \sum_{i=0}^{n-1} 2n + i$$

$$= 2P(H) + \frac{mn(3n - 1)}{2} + 2n^2 + n.$$
Let $h$ be a profile numbering of $G$. Now, assume that $P_h(G) < 2P(H) + mn(3n_i - 1) + 2n^2 + n$. For the same reason as in the case with $m = 2$, $h$ must completely number a row before starting another row. Furthermore, $h$ must begin with one of the end rows (say, $R_1$); otherwise $P_h(G) ≥ P_f(G) + n$. We claim that $h$ must then number the rows in the order $R_1; R_2; \ldots; R_{m-1}; R_m; R_{m-1}$.

We prove the claim by contradiction. Assume that this pattern is not followed. And, let the first violation of this pattern be for $h$ at $R_p$ to $R_q$, where $q = p + r$, $r > 1$ and $p < m - 1$. Then if $q < m - 1$,

$$\sum_{i=p+1}^{q-1} \sum_{j=1}^{\infty} w_h(v_{ij}) + P(H) + (5n^2_i - n) + \frac{(r \cdot 2)(3n^2_i \cdot n)}{2} > \frac{(r + 1)(3n^2_i \cdot n)}{2} = \sum_{i=p+1}^{q} \sum_{j=1}^{\infty} w_f(v_{ij}):$$

If $q = m - 1$,

$$\sum_{i=p+1}^{q-1} \sum_{j=1}^{\infty} w_h(v_{ij}) + P(H) + (5n^2_i - n) + \frac{(r \cdot 2)(3n^2_i \cdot n)}{2} > \frac{(r - 3)(3n^2_i \cdot n)}{2} = \sum_{i=p+1}^{q} \sum_{j=1}^{\infty} w_f(v_{ij}):$$

So, the claim is proved and $h$ must number the rows in the same order as $f$.

Since within each row $f$ numbers the same way as $g$, which is a profile numbering of $H$, it follows that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_f(v_{ij}) = 2P(H) + \frac{mn(3n_i - 1)}{2} + 2n^2 + n$$

$$< \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_h(v_{ij}) + \sum_{j=1}^{\infty} w_h(v_{mij}) + \frac{mn(3n_i - 1)}{2} i = 2P_h(G);$$

which implies that $P_f(G) < P_h(G)$, a contradiction.

Figure 1 shows a profile numbering for the graphs $P_4$, $C_5$, $K_{1;4}$, and $K_{2;3}$. In general, $P(P_n) = n_i \cdot 1$, $P(C_n) = 2n_i \cdot 3$, $P(K_n) = n(n_i \cdot 1) = 2$, $P(K_{1;n}) = n$, and
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for $r \cdot n$, $P(K_{r,n}) = rn + r(r - 1) = 2$ (see Lin and Yuan [4]). These observations lead directly to the following corollary.

Corollary 1. For $m \geq 3$;

\[
P(G) = \frac{mn(3n + 1)}{2} \cdot i \cdot 2n^2 + 3n \cdot i \cdot 2 \quad \text{f or } G = P_m[P_n];
\]

\[
P(G) = \frac{mn(3n + 1)}{2} \cdot i \cdot 2n^2 + 5n \cdot i \cdot 6 \quad \text{f or } G = P_m[C_n];
\]

\[
P(G) = \frac{mn(3n + 1)}{2} \cdot i \cdot n^2 \quad \text{f or } G = P_m[K_n];
\]

3. CYCLES AND COMPLETE GRAPHS WITH OTHER GRAPHS

In this section, we establish the profile of the composition of cycles with other graphs and the profile of the composition of complete graphs with other graphs.

Theorem 2. Let $G = C_m[H]$; where $H$ is a graph with $n$ vertices. Then

\[
P(G) = P(H) + \frac{n(5mn + 7n - m + 1)}{2};
\]

Proof. Let $g$ be a profile numbering of $H$. Define a numbering $f(v_{i,j}) = (i - 1)n + g(v_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then using an argument similar to that in the proof of Theorem 1, we see that a profile numbering is produced, namely,

\[
P_f(G) = P(H) + \frac{(m - 2)n(3n + 1)}{2} + \frac{n(2mn - n + 1)}{2};
\]

\[
P_f(G) = P(H) + \frac{n(5mn + 7n - m + 1)}{2};
\]

Corollary 2. For $m \geq 3$ and $n \geq 3$;

\[
P(G) = \frac{n(5mn + 7n - m + 3)}{2} \cdot 1 \quad \text{f or } G = C_m[P_n];
\]

\[
P(G) = \frac{n(5mn + 7n - m + 5)}{2} \cdot 3 \quad \text{f or } G = C_m[C_n];
\]

\[
P(G) = \frac{n(5mn + 6n - m)}{2} \quad \text{f or } G = C_m[K_n];
\]

Theorem 3. For $m \geq 1$; let $G = K_m[H]$; where $H$ is a graph with $n$ vertices. Then

\[
P(G) = P(H) + \frac{(mn + n - 1)(mn - n)}{2};
\]
Proof. In the composition of a complete graph with another graph, every vertex in $R_i$ is adjacent to all other vertices which are in rows other than $R_i$. Hence once we number a vertex 1, we know

$$X_{n}^{\infty} \sum_{i=n+1}^{n} w_i(v_i) = \frac{(mn + n \cdot 1)(mn \cdot n)}{2}.$$  

Since $\min_{i=1}^{\infty} w_i(v_i) = P(H)$, we then have $P(G) = P(H) + (mn + n \cdot 1)(mn \cdot n) = 2$; and $P(H) + (mn + n \cdot 1)(mn \cdot n) = 2$ is achievable by numbering $R_1$ with $1; \ldots ; n; R_2$ with $n + 1; \ldots ; 2n$ etc. and within each row following a profile numbering of $H$. So, $P(G) = P(H) + (mn + n \cdot 1)(mn \cdot n) = 2$.

Corollary 3. For $m \geq 1$ and $n \geq 1$;

$$P(K_m[K_n]) = \frac{mn(mn \cdot 1)}{2}.$$  

4. The Complete Bipartite Graph with Other Graphs

Theorem 4. Let $G = K_{mn}[H]$; where $m \cdot n$ and $H$ is a graph of order $l$.

Then

$$P(G) = mn l^2 + \frac{ml(ml \cdot 1)}{2} + nP(H):$$

Proof. Assume the two partite sets of vertices in $K_{mn}$ are $v_1; v_2; \ldots ; v_n; h$ and $v_{n+1}; v_{n+2}; \ldots ; v_{n+m}$. Also assume that $g$ is a profile numbering of $H$. Consider a numbering $f$ such that $f(v_{ij}) = g(v_i) + (ij - 1)$ for $1 \leq i \leq n + m, 1 \leq j \leq l$. Then

$$P_f(G) = nP(H) + \sum_{i=1}^{n} \sum_{j=1}^{l} f(v_{ij}) = mn l^2 + \frac{ml(ml \cdot 1)}{2} + nP(H):$$

Now assume that $h$ is a profile numbering of $G$. For $1 \leq i \leq n$ and $n + 1 \cdot j \leq n + m$, every vertex in $R_i$ is adjacent to every vertex in $R_j$, and for $m \cdot n$, the vertex $v = hi \cdot 1(1)$ should be in $R_i$. So

$$X_{n}^{\infty} \sum_{i=n+1}^{n} w_i(v_{ij}) = \frac{ml(ml \cdot 1)}{2}.$$  

Since \( \min_{i=1}^{p} \max_{i=1}^{p} w_{ij} = nP(H) \), we have \( P_n(G) = mn^2 + m(l(l+1)/2) + nP(H) \).

A direct application of Theorem 4 leads to Corollary 4.

**Corollary 4.**

\[
P(G) = \begin{cases} 
 mn^2 + nl + n + \frac{m(l(l+1))}{2} & \text{for } G = K_{m,n}[P_i], \\
 mn^2 + 2nl + 3n + \frac{m(l(l+1))}{2} & \text{for } G = K_{m,n}[C_i], \\
 mn^2 + \frac{nl(l+1)}{2} + \frac{m(l(l+1))}{2} & \text{for } G = K_{m,n}[K_i].
\end{cases}
\]

The profile of the composition of a star with any other graph also follows directly from Theorem 4.

**Corollary 5.** For \( n \geq 1 \) let \( H \) be a graph with \( l \) vertices. Then

\[
P(K_{1,n}[H]) = nl^2 + \frac{l(l+1)}{2} + nP(H).
\]

**Corollary 6.** For \( n \geq 1 \) and \( l \geq 1 \);

\[
P(G) = \begin{cases} 
 n^2 + nl + n + \frac{l(l+1)}{2} & \text{for } G = K_{1,n}[P_i], \\
 n^2 + 2nl + 3n + \frac{l(l+1)}{2} & \text{for } G = K_{1,n}[C_i], \\
 l^2 + \frac{(l+1)(n+1)}{2} & \text{for } G = K_{1,n}[K_i].
\end{cases}
\]

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**References**


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