**NULL 2-TYPE HYPERSURFACES WITH AT MOST THREE DISTINCT PRINCIPAL CURVATURES IN EUCLIDEAN SPACE**

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**Abstract.** The goal of this paper is to prove null 2-type hypersurfaces with at most three distinct principal curvatures in a Euclidean space have constant mean curvature.

1. **INTRODUCTION**

Let \( x : M^n \to \mathbb{E}^m \) be an isometric immersion of an \( n \)-dimensional connected submanifold \( M^n \) into a Euclidean space \( \mathbb{E}^m \). Denote by \( \Delta \) the Laplace operator with respect to the induced Riemannian metric. A submanifold of \( \mathbb{E}^m \) is said to be of finite type \([1, 2, 7, 9]\) if the position vector \( x \) of \( M^n \) in \( \mathbb{E}^m \) can be decomposed in the following form:

\[
x = x_0 + x_1 + \cdots + x_k,
\]

where \( x_0 \) is a constant vector and \( x_1, \ldots, x_k \) are non-constant maps satisfying \( \Delta x_i = \lambda_i x_i \), \( i = 1, \ldots, k \). In particular, if all eigenvalues \( \lambda_1, \ldots, \lambda_k \) are mutually different, then the submanifold \( M^n \) is said to be of \( k \)-type and if one of \( \lambda_1, \ldots, \lambda_k \) is zero, \( M^n \) is said to be of null \( k \)-type.

We now focus on null 2-type submanifolds \( M^n \) in \( \mathbb{E}^m \). By choosing a coordinate system on \( \mathbb{E}^m \) with \( x_0 \) as its origin, we have the following simple spectral decomposition of \( x \) for a null 2-type submanifold \( M^n \):

\[
x = x_1 + x_2, \quad \Delta x_1 = 0, \quad \Delta x_2 = ax_2,
\]

where \( a \) is non-zero constant. After applying Beltrami’s formula \( \Delta x = -n\overrightarrow{H} \), where \( \overrightarrow{H} \) is the mean curvature vector, (1.2) implies the following equation

\[
\Delta \overrightarrow{H} = a \overrightarrow{H}.
\]

Chen proposed in 1991 the following interesting problem [2, Problem 12]:

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"Determine all submanifolds of Euclidean spaces which are of null 2-type. In particular, classify null 2-type hypersurfaces in Euclidean spaces."

In 1988, Chen [3] firstly proved that a null 2-type surface in \( E^3 \) is an open portion of a circular cylinder \( S^1 \times \mathbb{R} \). Later on, Ferrández and Lucas [14] generalized Chen’s results by showing that a null 2-type Euclidean hypersurface in \( E^{n+1} \) with at most two distinct principal curvatures is a spherical cylinder \( S^p \times \mathbb{R}^{n-p} \). In 1995, Hasanis and Vlachos [15] proved that null 2-type hypersurfaces in \( E^5 \) have constant mean curvature (see also Defever’s proof in [11]). Recently, Chen and Garay in [8] characterized \( \delta(2) \)-ideal null 2-type hypersurfaces in Euclidean space as spherical cylinders, where \( \delta(2) \)-ideal hypersurfaces are a class of hypersurfaces whose principal curvatures take three special values: \( \eta, \mu \) and \( \eta + \mu \). There are also some study on null 2-type submanifolds with codimension greater one due to U. Dursun ([12, 13]). For more work in this field, see Chen’s recent excellent survey [10].

A remarkable property obtained by Chen [4] says that a submanifold \( M^n \) of Euclidean space satisfies (1.3) if and only if \( M^n \) is 1) Biharmonic (in this case, \( a = 0 \)); 2) 1-type; 3) null 2-type.

As pointed out by Chen et al., for example, in [8], a 1-type submanifold of a Euclidean space \( E^m \) is either a minimal submanifold of \( E^m \) or a minimal submanifold of a hypersphere in \( E^m \). Biharmonic submanifolds in \( E^m \) are defined by the equation \( \Delta \overline{H} = 0 \), which is equivalent to \( \Delta^2 x = 0 \). Chen [2] in 1991 stated a well-known conjecture: The only biharmonic submanifolds of Euclidean spaces are the minimal ones. This conjecture is still open so far and the study of biharmonic submanifolds is a very active field [10].

In this paper, we investigate null 2-type hypersurfaces with at most three distinct principal curvatures in Euclidean space. Precisely, we will prove that

**Theorem 1.1.** Every null 2-type hypersurface with at most three distinct principal curvatures in a Euclidean space must have constant mean curvature.

Remark that our result generalizes the results given in [3, 8, 14, 15].

2. Preliminaries

Let \( x : M^n \rightarrow E^{n+1} \) be an isometric immersion of a hypersurface \( M^n \) into \( E^{n+1} \). Denote the Levi-Civita connections of \( M^n \) and \( E^{n+1} \) by \( \nabla \) and \( \tilde{\nabla} \), respectively. Let \( X \) and \( Y \) denote vector fields tangent to \( M^n \) and let \( \xi \) be a unite normal vector field. Then the Gauss and Weingarten formulas are given, respectively, by (cf. [5, 6])

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}
\]

\[
\tilde{\nabla}_X \xi = -AX, \tag{2.2}
\]
where \( h \) is the second fundamental form, and \( A \) is the shape operator (or Weingarten operator). It is well known that the second fundamental form \( h \) and the shape operator \( A \) are related by

\[
\langle h(X, Y), \xi \rangle = \langle A \xi, X \rangle.
\]

The mean curvature vector \( \overrightarrow{H} \) is given by

\[
\overrightarrow{H} = \frac{1}{n} \text{trace } h.
\]

The Gauss and Codazzi equations are given respectively by

\[
R(X, Y)Z = \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,
\]

\[
(\nabla_X A)Y = (\nabla_Y A)X,
\]

for all \( X, Y, Z \) tangent to \( M \).

Assume that \( \overrightarrow{H} = H \xi \). Note that \( H \) denotes the mean curvature. By identifying the tangent and the normal parts of the condition \( \Delta \overrightarrow{H} = a \overrightarrow{H} \) \((a \neq 0)\), we obtain necessary and sufficient conditions for \( M^n \) to be of null 2-type in \( \mathbb{E}^{n+1} \).

**Proposition 2.1.** Assume \( M^n \) is not 1-type. A hypersurface \( M^n \) in an \( n + 1 \)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) is null 2-type if and only if

\[
\begin{align*}
\Delta H + H \text{trace } A^2 &= aH, \\
2A \text{grad}H + nH \text{grad}H &= 0,
\end{align*}
\]

where the Laplace operator \( \Delta \) acting on scalar-valued function \( f \) is given by (e.g., [8])

\[
\Delta f = -\sum_{i=1}^{n}(e_i e_i f - \nabla_{e_i}e_i f).
\]

Here, \( \{e_1, \ldots, e_n\} \) is an orthonormal local tangent frame on \( M^n \).

3. PROOF OF THEOREM 1.1

In what follows, we work on null 2-type hypersurfaces \( M^n \) with three distinct principal curvatures in Euclidean space \( \mathbb{E}^{n+1} \) with \( n \geq 4 \).

Suppose that the mean curvature \( H \) is not constant. We will derive a contradiction.
By the second equation of (2.6), it is easy to see that $\text{grad } H$ is an eigenvector of the Weingarten operator $A$ with the corresponding principal curvature $-\frac{1}{2} H$. Without loss of generality, we choose $e_1$ such that $e_1$ is parallel to $\text{grad } H$, and therefore the Weingarten operator $A$ of $M^n$ takes the following form with respect to a suitable orthonormal frame $\{e_1, \ldots, e_n\}$.

(3.1) \[
A = \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{pmatrix},
\]

where $\lambda_i$ are the principal curvatures and $\lambda_1 = -\frac{1}{2} H$. Since $e_1$ is parallel to $\text{grad } H$, we compute

\[
\text{grad } H = \sum_{i=1}^{n} e_i(H)e_i
\]

and hence

(3.2) \[
e_1(H) \neq 0, \quad e_i(H) = 0, \quad i = 2, 3, \ldots, n.
\]

We write

(3.3) \[
\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \quad i, j = 1, 2, \ldots, n.
\]

We compute $\nabla_{e_k} \langle e_i, e_i \rangle = 0$ and $\nabla_{e_k} \langle e_i, e_j \rangle = 0$, which imply respectively that

(3.4) \[
\omega_{ki}^k = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0,
\]

for $i \neq j$ and $i, j, k = 1, 2, \ldots, n$. Furthermore, we deduce from (3.1) and (3.3) and the Codazzi equation that

(3.5) \[
e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j,
\]

(3.6) \[
(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j
\]

for distinct $i, j, k = 1, 2, \ldots, n$.

It follows from (3.2) and (3.3) that

\[
[e_i, e_j](H) = 0, \quad i, j = 2, 3, \ldots, n, \quad i \neq j,
\]

which yields

(3.7) \[
\omega_{ij}^1 = \omega_{ji}^1,
\]
Null 2-Type Hypersurfaces with Three Distinct Principal Curvatures

for distinct \( i, j = 2, 3, \ldots, n \).

We claim that \( \lambda_j \neq \lambda_1 \) for \( j = 2, 3, \ldots, n \). In fact, if \( \lambda_j = \lambda_1 \) for \( j \neq 1 \), by putting \( i = 1 \) in (3.5) we have that

\[
0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1),
\]

which contradicts to the first expression of (3.2).

By the assumption, \( M^n \) is a nondegenerate hypersurface with three distinct principal curvatures. Without loss of generality, we assume that

\[
\lambda_2 = \lambda_3 = \cdots = \lambda_p = \alpha,
\]

\[
\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n = \beta
\]

for \( \frac{p+1}{2} \leq p < n \). The multiplicities of principal curvatures \( \alpha \) and \( \beta \) are \( p - 1 \) and \( n - p \), respectively.

By the definition (2.4) of \( \overrightarrow{H} \), we have \( nH = \sum_{i=1}^{n} \lambda_i \). Hence

\[
\beta = \frac{\frac{3}{2}nH - (p-1)\alpha}{n - p}.
\]

Hence, by \( \lambda_1 = -\frac{n}{2}H \) and (3.9), \( \alpha \neq \lambda_1, \beta \) and \( \beta \neq \lambda_1 \) yield directly that

\[
\alpha \neq -\frac{n}{2}H, \quad \beta = \frac{\frac{3}{2}nH - (p-1)\alpha}{n - p}, \quad \frac{n^2 - (p-3)n}{2(p-1)}H.
\]

Since \( n \geq 4 \), it follows from (3.9) that \( p-1 \geq 2 \). For \( i, j = 2, 3, \ldots, p \) and \( i \neq j \) in (3.5), one has

\[
e_{i}(\alpha) = 0, \quad i = 2, 3, \ldots, p.
\]

Depending on the multiplicity \( n - p \) of the principal curvature \( \beta \), we consider two cases:

**Case A.** \( n - p \geq 2 \). In this case, for \( i, j = p+1, \ldots, n \) and \( i \neq j \) in (3.5) we have

\[
e_i(\beta) = 0, \quad i = p+1, \ldots, n.
\]

Hence, it follows directly from (3.2), (3.9), (3.11) and (3.12) that

\[
e_i(\alpha) = 0, \quad i = 2, \ldots, n.
\]

**Case B.** \( n - p = 1 \). Then (3.11) reduces to

\[
e_i(\alpha) = 0, \quad i = 2, \ldots, n - 1.
\]
In this case, we will show that $e_{n}(\alpha) = 0$ in the following.

Let us compute $[e_1, e_i](H) = (\nabla_{e_1} e_i - \nabla_{e_i} e_1)(H)$ for $i = 2, \ldots, n$. From the first expression of (3.4), we have $\omega_{1i}^1 = 0$. For $j = 1$ and $i \neq 1$ in (3.5), by (3.2) we have $\omega_{1i}^1 = 0 \ (i \neq 1)$. Hence we have

$$e_i e_1(H) = 0, \quad i = 2, \ldots, n. \tag{3.15}$$

By (3.14), with a similar way we can show that

$$e_i e_1(\alpha) = 0, \quad i = 2, \ldots, n - 1. \tag{3.16}$$

For $j = 1, k, i \neq 1$ in (3.6) we have

$$(\lambda_i - \lambda_1)\omega_{ki}^1 = (\lambda_k - \lambda_1)\omega_{ik}^1,$$

which together with (3.7) yields

$$\omega_{ij}^1 = 0, \quad i \neq j, \quad i, j = 2, \ldots, n. \tag{3.17}$$

Combining (3.17) with the second equation of (3.4) gives

$$\omega_{i1}^j = 0, \quad i \neq j, \quad i, j = 2, \ldots, n. \tag{3.18}$$

It follows from (3.5) that

$$\omega_{i1}^i = \frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i}, \quad i = 2, \ldots, n. \tag{3.19}$$

For $k = 2$ and $i = n$ in (3.6), we have

$$(\lambda_n - \lambda_j)\omega_{2n}^j = (\lambda_2 - \lambda_j)\omega_{n2}^j,$$

which yields

$$\omega_{2n}^j = 0, \quad j = 3, \ldots, n - 1.$$  

Hence, from the first expression of (3.4) and (3.17) we get

$$\omega_{2n}^j = 0, \quad j = 1, 3, \ldots, n. \tag{3.20}$$

Also, (3.5) yields

$$\omega_{2n}^2 = \frac{e_n(\alpha)}{\lambda_n - \alpha}. \tag{3.21}$$

In the following we will derive a useful equation.
From the Gauss equation and (3.1) we have \( R(e_2, e_n)e_1 = 0 \). Recall the definition of Gauss curvature tensor

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.
\]

It follows from (3.16), (3.18-21) and (3.4) that

\[
\nabla_{e_2} \nabla_{e_n} e_1 = \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} e_2,
\]

\[
\nabla_{e_n} \nabla_{e_2} e_1 = e_n(e_1(\lambda_n)/(\lambda_1 - \alpha) + \frac{e_1(\alpha)}{\lambda_1 - \alpha} \sum_{k=3}^{n} \omega_{n2}^k e_k),
\]

\[
\nabla_{[e_2, e_n]} e_1 = \frac{e_n(\alpha)e_1(\alpha)}{(\lambda_n - \alpha)(\lambda_1 - \alpha)} e_2 - \frac{e_1(\alpha)}{\lambda_1 - \alpha} \sum_{k=3}^{n} \omega_{n2}^k e_k.
\]

Hence

\[
e_n(\frac{e_1(\alpha)}{\lambda_1 - \alpha}) = \frac{e_1(\lambda_n)/\lambda_1 - \alpha - e_1(\alpha)/\lambda_1 - \alpha}{\lambda_1 - \lambda_n}) e_n(\alpha).
\]

Note that \( \lambda_1 = -\frac{n}{2}H \) and \( \lambda_n = \beta = \frac{3}{2}nH - (n - 2)\alpha \).

Equation (3.22) can be rewritten as

\[
e_n e_1(\alpha) = \{ - \frac{e_1(\alpha)}{\lambda_1 - \alpha} + \frac{e_1(\lambda_n)/\lambda_1 - \lambda_n - e_1(\alpha)/\lambda_1 - \alpha}{\lambda_1 - \lambda_n}) e_n(\alpha),
\]

and hence

\[
e_n(\frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n}) = -(n - 2)(\frac{e_n e_1(\alpha)}{\lambda_1 - \lambda_n} + \frac{e_1(\lambda_n)e_n(\alpha)}{(\lambda_1 - \lambda_n)^2})
\]

\[
= -(n - 2)\frac{e_n(\alpha)}{\lambda_1 - \lambda_n} \frac{e_1(\lambda_n)/\lambda_1 - \lambda_n - e_1(\alpha)/\lambda_1 - \alpha}{\lambda_1 - \lambda_n}) e_1(\alpha) \frac{\lambda_1 + \lambda_n - 2\alpha}{\lambda_n - \alpha}.
\]

Consider the first equation of (2.6). It follows from (3.1) and (3.19) that

\[
e_1 e_1(H) + \frac{(n - 2)e_1(\alpha)}{\lambda_1 - \alpha} + \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n}) e_1(\alpha) = -(\frac{\lambda_1^2 + (n - 2)\alpha^2 + \lambda_n^2}{\lambda_1 - \alpha}) = -aH.
\]

From (3.15) and \( \omega_{1n}^1 = \omega_{n1}^1 = 0 \), by computing \([e_1, e_n](e_1(H)) = (\nabla e_1 e_n - \nabla e_n e_1)(e_1(H)) = 0\), we could deduce that \( e_n(e_1 e_1(H)) = 0 \).

Now differentiating (3.24) along \( e_1 \), by (3.2), (3.15), (3.22) and (3.23) we get

\[
\frac{2}{\lambda_1 - \lambda_n} \frac{e_1(\lambda_n)/\lambda_1 - \lambda_n - \alpha}{\lambda_1 - \alpha}) e_1(H) e_n(\alpha) + H(-3nH + 2(n - 1)\alpha)e_n(\alpha) = 0.
\]
If $e_n(\alpha) \neq 0$, then the above equation becomes

\begin{equation}
\frac{2}{\lambda_1 - \lambda_n} \left( \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{\alpha}{\lambda_1 - \alpha} \right) e_1(H) + H(-3nH + 2(n-1)\alpha) = 0.
\end{equation}

Differentiating (3.25) along $e_n$, using (3.22) and (3.23) one has

\begin{equation}
\frac{2n(4-n)H + 2(n-2)(n-1)\alpha}{(\lambda_1 - \lambda_n)(\lambda_n - \alpha)} \left( \frac{e_1(\lambda_n)}{\lambda_1 - \lambda_n} - \frac{\alpha}{\lambda_1 - \alpha} \right) e_1(H)
+ H((-7n+10)nH + 4(n-1)(n-2)\alpha) = 0.
\end{equation}

Therefore, combining (3.26) with (3.25) gives

\begin{equation}
3(n-2)H(3nH - 2(n-1)\alpha)^2 = 0,
\end{equation}

which implies that

\begin{equation}
\alpha = \frac{3n}{2(n-1)}H.
\end{equation}

This contradicts to (3.10). Hence, we have that $e_n(\alpha) = 0$. Now we are ready to express the connection coefficients of hypersurfaces.

**Lemma 3.1.** Under the assumptions above, we have

\[\nabla_{e_i} e_1 = 0; \quad \nabla_{e_i} e_1 = \frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_i, \quad i = 2, \ldots, n;\]

\[\nabla_{e_i} e_j = \sum_{k=2, k \neq j}^{p} \omega_{ij}^k e_k, \quad i = 1, \ldots, n, \quad j = 2, \ldots, p, \quad i \neq j;\]

\[\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=2, k \neq i}^{p} \omega_{ii}^k e_k, \quad i = 2, \ldots, p;\]

\[\nabla_{e_i} e_j = \sum_{k=p+1, k \neq j}^{n} \omega_{ij}^k e_k, \quad i = 1, \ldots, n, \quad j = p+1, \ldots, n, \quad i \neq j;\]

\[\nabla_{e_i} e_i = -\frac{e_1(\lambda_i)}{\lambda_1 - \lambda_i} e_1 + \sum_{k=p+1, k \neq i}^{n} \omega_{ii}^k e_k, \quad i = p+1, \ldots, n.\]

**Proof.** For $j = 1$ and $i = 2, \ldots, n$ in (3.5), by (3.2) we get $\omega_{1i}^1 = 0$. Moreover, by the first and second expressions of (3.4) we have

\begin{equation}
\omega_{1i}^1 = \omega_{11}^i = 0, \quad i = 1, \ldots, n.
\end{equation}
For $i = 1$, $j = 2, \ldots, n$ in (3.5), we obtain
\[(3.28) \quad \omega^j_{j1} = -\omega^1_{jj} = \frac{e_1(\lambda_j)}{\lambda_1 - \lambda_j}, \quad j = 2, \ldots, n.\]

For $i = p + 1, \ldots, n$, $j = 2, \ldots, p$ in (3.5), by (3.2) we have
\[(3.29) \quad \omega^j_{ji} = -\omega^i_{jj} = 0.\]

Similarly, for $i = 2, \ldots, p$, $j = p + 1, \ldots, n$ in (3.5), we also have
\[(3.30) \quad \omega^j_{ji} = -\omega^i_{jj} = 0.\]

For $i = 1$, by choosing $j, k = 2, \ldots, p$ or $k, j = p + 1, \ldots, n$ ($j \neq k$) in (3.6), we have
\[(3.31) \quad \omega^j_{k1} = \omega^1_{kj} = 0.\]

For $i = 2, \ldots, p$ and $j, k = p + 1, \ldots, n$ ($j \neq k$) in (3.6), we get
\[(3.32) \quad \omega^j_{ki} = \omega^k_{ji} = 0.\]

For $i = 2, \ldots, p$, $j = 1$ and $k = p + 1, \ldots, n$ in (3.6), one has
\[(\alpha - \lambda_1)\omega^1_{ki} = (\beta - \lambda_1)\omega^i_{k1},\]
which together with (3.7) and the second expression of (3.4) gives
\[(3.33) \quad \omega^1_{ki} = \omega^1_{ik} = \omega^i_{k1} = \omega^k_{i1} = 0.\]

For $i = 2, \ldots, p$, $k = 1$ and $j = p + 1, \ldots, n$ in (3.6), we obtain
\[(\beta - \alpha)\omega^j_{1i} = (\lambda_1 - \alpha)\omega^j_{i1},\]
which together with (3.33) yields
\[(3.34) \quad \omega^j_{1i} = \omega^j_{i1} = 0.\]

Combining (3.27-3.34) with (3.4) completes the proof of the lemma.

Define two smooth functions $A$ and $B$ as follows:
\[(3.35) \quad A = \frac{e_1(\alpha)}{\lambda_1 - \alpha}, \quad B = \frac{e_1(\beta)}{\lambda_1 - \beta}.\]

One can compute the curvature tensor $R$ by Lemma 3.1, and apply the Gauss equation for different values of $X$, $Y$, and $Z$. After comparing the coefficients with respect to the orthonormal basis $\{e_1, \ldots, e_n\}$ we get the following:
\( X = e_1, Y = e_2, Z = e_1, \)

\[
e_1(A) + A^2 = -\lambda_1 \alpha;
\]

\( X = e_1, Y = e_n, Z = e_1, \)

\[
e_1(B) + B^2 = -\lambda_1 \beta;
\]

\( X = e_n, Y = e_2, Z = e_n, \)

\[
AB = -\alpha \beta.
\]

Note that equation (3.38) can be obtained by calculating \( \langle R(e_n, e_2)e_n, e_2 \rangle \).

Compute the first equation of (2.6) again. It follows from (3.1) and Lemma 3.1 that

\[
e_1 e_1(H) - \left\{ (p - 1) A + (n - p) B \right\} e_1(H) + H(\lambda_1^2
\]

\[
+ (p - 1) \alpha^2 + (n - p) \beta^2) = aH.
\]

**Lemma 3.2.** The functions \( A \) and \( B \) are related by

\[
\left\{ (4 - p) A + (3 + p - n) B \right\} e_1(H) + \frac{3n^2(n + 6 - p)}{4(n - p)} H^3
\]

\[
- \frac{3n(n - 2 + 4p)}{2(n - p)} H^2 \alpha + \frac{3n(p - 1)}{n - p} H \alpha^2 - \frac{3}{2} aH = 0.
\]

**Proof.** From (3.35), (3.36) and (3.37) respectively reduce to

\[
e_1 e_1(\alpha) + 2A e_1(\alpha) - A e_1(\lambda_1) + \lambda_1 \alpha (\lambda_1 - \alpha) = 0,
\]

\[
e_1 e_1(\beta) + 2B e_1(\beta) - B e_1(\lambda_1) + \lambda_1 \beta (\lambda_1 - \beta) = 0.
\]

By (3.9), it follows from the second expression of (3.35) that

\[
e_1(\alpha) = \frac{3n}{2(p - 1)} e_1(H) - \frac{n - p}{p - 1} B(\lambda_1 - \beta).
\]

Similarly,

\[
e_1(\beta) = \frac{3n}{2(n - p)} e_1(H) - \frac{p - 1}{n - p} A(\lambda_1 - \alpha).
\]

Substitute (3.9) into (3.42). Eliminating \( e_1 e_1(H) \) and \( e_1 e_1(\alpha) \), from (3.38), (3.39) and (3.41-44) we obtain the desired equation (3.40).
Now we are in a position to prove Theorem 1.1.

Proof. By the second expression of (3.35) and (3.9), equation (3.44) reduces to

\[ e_1(H) = -\left\{ \frac{p - 1}{3} H + \frac{2(p - 1)}{3n} \alpha \right\} A + \left\{ -\frac{n + 3 - p}{3} H + \frac{2(p - 1)}{3n} \alpha \right\} B. \tag{3.45} \]

Substituting (3.45) into (3.40), by (3.38) we have

\[ (4 - p)(p - 1)(nH + 2\alpha)A^2 + (3 + p - n)\{ n(n + 3 - p)H - 2(p - 1)\alpha \} B^2 = f(H, \alpha), \tag{3.46} \]

where

\[ f(H, \alpha) = \frac{9n^3(n + 6 - p)}{4(n - p)} H^3 + \frac{3n^2(p - 1)(2p - 2n - 15)}{2(n - p)} H^2 \alpha \\
+ \frac{n(p - 1)(-2p^2 + 2pn + 11p + n - 12)}{n - p} H \alpha^2 \\
- \frac{2(p - 1)^2(2p - n - 1)}{n - p} \alpha^3 - \frac{9}{2} nH. \tag{3.47} \]

Multiplying \( A \) and \( B \) successively on the equation (3.40), using (3.38) one gets respectively

\[ (4 - p)A^2 e_1(H) - (3 + p - n)\alpha e_1(H) \]
\[ + \left\{ \frac{3n^2(n + 6 - p)}{4(n - p)} H^3 - \frac{3n(n - 2 + 4p)}{2(n - p)} H^2 \alpha + \frac{3n(p - 1)}{n - p} H \alpha^2 - \frac{3}{2} nH \right\} A = 0, \tag{3.48} \]
\[ (3 + p - n)B^2 e_1(H) - (4 - p)\alpha e_1(H) \]
\[ + \left\{ \frac{3n^2(n + 6 - p)}{4(n - p)} H^3 - \frac{3n(n - 2 + 4p)}{2(n - p)} H^2 \alpha + \frac{3n(p - 1)}{n - p} H \alpha^2 - \frac{3}{2} nH \right\} B = 0. \tag{3.49} \]

Differentiating (3.40) along \( e_1 \), and using (3.36-37) and (3.39) we get

\[ \left\{ (4 - p)\left( \frac{n}{2} H \alpha - A^2 \right) + (3 + p - n)\left( \frac{n}{2} H \beta - B^2 \right) \right\} e_1(H) \]
\[ - \left\{ (4 - p)A + (3 + p - n)B \right\} \left\{ (p - 1)A + (n - p)B \right\} e_1(H) \]
\[ + \left\{ (4 - p)A + (3 + p - n)B \right\} \left\{ \frac{n^2}{4} H^3 + (p - 1)H \alpha^2 + (n - p)H \beta^2 - nH \right\} \]
\[ + \left\{ \frac{9n^2(n + 6 - p)}{4(n - p)} H^2 - \frac{3n(n - 2 + 4p)}{n - p} H \alpha + \frac{3n(p - 1)}{n - p} \alpha^2 - \frac{3}{2} \right\} e_1(H) \]
\[ - \left\{ \frac{3n(n - 2 + 4p)}{2(n - p)} H^2 e_1(\alpha) + \frac{6n(p - 1)}{n - p} H \alpha e_1(\alpha) \right\} = 0. \tag{3.50} \]
Substituting (3.40), (3.47), (3.48) into (3.49), and using the first expression of (3.35) we obtain

\[
\begin{align*}
\left\{ \frac{3n^2(2n-2p+21)}{4(n-p)}H^2 - \frac{3n(5p+1)}{n-p}H\alpha + \frac{(p-1)(2n+7)}{n-p}\alpha^2 - \frac{3}{2}a \right\}e_1(H) \\
+ \left\{ \frac{n^2(2pn-2p^2+7n+17p+30)}{4(n-p)}H^3 - \frac{3n(3np+2p^2+4p-3n-6)}{2(n-p)}H^2\alpha \\
+ \frac{(p-1)(2np-2n+p-4)}{n-p}H\alpha^2 + \frac{1}{2}(5p-8)\alpha a \right\}A \\
+ \left\{ \frac{n^2(2(n-p)^2+15(n-p)+45)}{4(n-p)}H^3 - \frac{3n(n^2+np-2p^2+10p+n-8)}{2(n-p)}H^2\alpha \\
+ \frac{(p-1)(2n^2-2np+7n-p-3)}{n-p}H\alpha^2 + \frac{1}{2}(5n-5p-3)\alpha H \right\}B = 0.
\end{align*}
\]

Moreover, it follows from (3.45) that the above equation further reduces to

\[
(3.51) \quad L(H,\alpha)A + M(H,\alpha)B = 0,
\]

where

\[
L(H,\alpha) = \frac{2n^3(3n-2p+17)}{4}H^3 - \frac{2n^2(-6p^2+11np+43p-11n-37)}{2}H^2\alpha
\]

\[
+ n(p-1)(4np-4n+26p+1)H\alpha^2 - 2(p-1)^2(2n+7)\alpha^3 \\
+ \frac{2n(n-p)(2p-3)\alpha a}{2(n-p)(p-1)\alpha a},
\]

\[
M(H,\alpha) = -\frac{2}{2}(2n-2p+3)H^3 - \frac{2n^2(2p^2+n^2-3np-7p+n-3)}{2}H^2\alpha
\]

\[
+ 2n(p-1)(2n^2-2np+4n-13p-18)H\alpha^2 + 2(p-1)^2(2n+7)\alpha^3 \\
- 9n(n-p)^2\alpha aH + 3(n-p)(p-1)\alpha a.
\]

Multiplying \(LM\) on the equation (3.46), using (3.51-3.53) and (3.38) we can eliminate both \(A\) and \(B\). Hence, we have

\[
(4-p)(p-1)(nH+2H)M^2\frac{2nH\alpha - (p-1)\alpha^2}{n-p}
\]

\[
+(3+p-n)\left\{ n(n+3-p)H - 2(p-1)\alpha \right\}L^2\frac{2nH\alpha - (p-1)\alpha^2}{n-p}
\]

\[
LMf = 0.
\]

In view of (3.54), we notice that the equation should take the following form:

\[
\begin{align*}
c_{90}H^9 + c_{81}H^8\alpha + c_{72}H^7\alpha^2 + c_{63}H^6\alpha^3 + c_{54}H^5\alpha^4 + c_{45}H^4\alpha^5 \\
c_{36}H^3\alpha^6 + c_{27}H^2\alpha^7 + c_{18}H\alpha^8 + c_{09}\alpha^9 + a(c_{70}H^7 + c_{61}H^6) \\
c_{52}H^5\alpha^2 + c_{43}H^4\alpha^3 + c_{34}H^3\alpha^4 + c_{25}H^2\alpha^5 + c_{16}H\alpha^6 + c_{07}\alpha^7 \\
c_{50}H^5 + c_{41}H^4\alpha + c_{32}H^3\alpha^2 + c_{23}H^2\alpha^3 + c_{14}H\alpha^4 + c_{05}\alpha^5 \\
c_{30}H^3 + c_{21}H^2\alpha + c_{12}H\alpha^2 + c_{03}\alpha^3 = 0,
\end{align*}
\]

Yu Fu
where the coefficients $c_{ij}$ ($i, j = 0, \ldots, 9$) are constants concerning $n$ and $p$.

From (3.54), (3.52), (3.53) and (3.47), we compute $a_{90}$ as follows

$$c_{90} = \frac{729n^6(n - p + 6)(3n - 2p + 17)(2n - 2p + 3)}{32(n - p)}.$$  

Since $n > p$, it is easy to see that $c_{90} \neq 0$.

Note that $\alpha$ is not constant in general. In fact, if $\alpha$ is a constant, then (3.55) becomes an algebraic equation of $H$ with constant coefficients. Thus, the real function $H$ satisfies a polynomial equation $q(H) = 0$ with constant coefficients, therefore it must be a constant. We obtain the conclusion immediately.

Now consider an integral curve of $e_1$ passing through $p = \gamma(t_0)$ as $\gamma(t), t \in I$. Since $e_i(H) = e_i(\alpha) = 0$ for $i = 2, \ldots, n$ and $e_1(H), e_1(\alpha) \neq 0$, we can assume $t = t(\alpha)$ and $H = H(\alpha)$ in some neighborhood of $\alpha_0 = \alpha(t_0)$.

From the first expression of (3.35), (3.45) and (3.51), we have

$$\frac{dH}{d\alpha} = \frac{dH}{dt} \frac{dt}{d\alpha} = \frac{e_1(H)}{e_1(\alpha)}$$

(3.56)

$$= \frac{2(p - 1) + \frac{2(p - 1)}{3n} A \alpha - \frac{2(p - 1)}{3n} A B}{-\frac{3}{2} H - \alpha A}$$

$$= \frac{2(p - 1) + \frac{2(p - 1)}{3n} A B}{-\frac{3}{2} H - \alpha A}.$$  

Differentiating (3.55) with respect to $\alpha$ and substituting $\frac{dH}{d\alpha}$ from (3.56), combining these with (3.51) we get another algebraic equation of twelfth degree concerning $H$ and $\alpha$

$$b_{12,0} H^{12} + b_{11,1} H^{11} \alpha + b_{10,2} H^{10} \alpha^2 + b_{93} H^9 \alpha^3 + b_{84} H^8 \alpha^4 + b_{75} H^7 \alpha^5 + b_{66} H^6 \alpha^6 + b_{57} H^5 \alpha^7 + b_{48} H^4 \alpha^8 + b_{39} H^3 \alpha^9 + b_{210} H^2 \alpha^{10} + b_{111} H \alpha^{11} + b_{0,12} \alpha^{12} + c(b_{10,0} H^{10} + b_{91} H^9 \alpha + b_{82} H^8 \alpha^2 + b_{73} H^7 \alpha^3 + b_{64} H^6 \alpha^4$$

(3.57)  

$$+ b_{55} H^5 \alpha^5 + b_{46} H^4 \alpha^6 + b_{37} H^3 \alpha^7 + b_{28} H^2 \alpha^8 + b_{19} H \alpha^9 + b_{0,10} \alpha^{10} + b_{00} H^8 + b_{71} \alpha^{11} + b_{62} H^6 \alpha^2 + b_{53} H^5 \alpha^3 + b_{44} H^4 \alpha^4 + b_{35} H^3 \alpha^5 + b_{26} H^2 \alpha^6 + b_{17} H \alpha^7 + b_{08} \alpha^8 + b_{69} H^6 + b_{51} H^5 \alpha + b_{42} H^4 \alpha^2 + b_{33} H^3 \alpha^3 + b_{24} H^2 \alpha^4 + b_{15} H \alpha^5 + b_{06} \alpha^6 + b_{40} H^4 + b_{31} H^3 \alpha + b_{22} H^2 \alpha^2 + b_{13} H \alpha^3 + b_{04} \alpha^4 = 0,$$

where the coefficients $b_{ij}$ ($i, j = 0, \ldots, 12$) are constants concerning $n$ and $p$.

Note that equation (3.57) is non-trivial and different from (3.55).

We rewrite (3.55) and (3.57) respectively in the following forms
\[ \sum_{i=0}^{9} q_i(H) \alpha^i = 0, \quad \sum_{j=0}^{12} \bar{q}_j(H) \alpha^j = 0, \]

where \( q_i(H) \) and \( \bar{q}_j(H) \) are polynomials concerning function \( H \).

We may eliminate \( \alpha \) between the two equations of (3.58). Multiplying \( \bar{q}_{12}(H) \alpha^3 \) and \( q_8(H) \) respectively on the first and second equations of (3.58), we obtain a new polynomial equation of \( \alpha \) with eleventh degree. Combining this equation with the first equation of (3.58), we successively obtain a polynomial equation of \( \alpha \) with tenth degree. In a similar way, by using the first equation of (3.58) and its consequences we are able to gradually eliminate \( \alpha \).

At last, we obtain a non-trivial algebraic polynomial equation of \( H \) with constant coefficients. Therefore, we conclude that the real function \( H \) must be a constant, which contradicts our original assumption.

In conclusion, we complete the proof of Theorem 1.1.

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