ENDOMORPHISM RINGS OF MODULES OVER PRIME RINGS

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Abstract. Endomorphism rings of modules appear as the center of a ring, as the fix ring of a ring with group action or as the subring of constants of a derivation. This note discusses the question whether certain \( {*} \)-prime modules have a prime endomorphism ring. Several conditions are presented that guarantee the primeness of the endomorphism ring. The contours of a possible example of a \( {*} \)-prime module whose endomorphism ring is not prime are traced.

1. INTRODUCTION

Endomorphism rings of modules appear in many ring theoretical situations. For example the center \( C(R) \) of a (unital, associative) ring \( R \) is isomorphic to the endomorphism ring of \( R \) seen as a bimodule over itself, i.e. as a left \( R \otimes R^{op} \)-module. The subring \( R^G \) of elements that are left invariant under the action of a group \( G \) on \( R \) is isomorphic to the endomorphism ring of \( R \) seen as a left module over the skew group ring \( R \ast G \). The subring \( R^\partial \) of constants of a derivation \( \partial \) of \( R \) is isomorphic to the endomorphism ring of \( R \) seen as a left module over its differential operator ring \( R[x, \partial] \). More generally the subring \( R^H \) of elements invariant under the action of a Hopf algebra \( H \) acting on \( R \) is isomorphic to the endomorphism ring of \( R \) seen as left module over the smash product \( R\# H \). This identifications motivated the use of module theory in the study of Hopf algebra actions in [4, 10, 11, 12].

Prime numbers and prime ideals are basic concepts in algebra. While the idea of a prime ideal is well established, the idea of a prime submodule of a module is not. The purely essence of a prime ideal had been distilled already by Birkhoff in the concept...
of a prime element in a partially ordered groupoid. In [3], Bican et al. introduced an operation on the lattice of submodules of a module, turning it into a partially ordered groupoid. Let \( R \) be any (associative, unital) ring and \( M \) a left \( R \)-module. For any submodules \( N, L \) we denote

\[
N \ast L = N\text{Hom}(M, L) = \sum\{(N)f \mid f : M \to L\}.
\]

Note that we will write homomorphisms opposite of scalars, i.e. on the right side of an element. A submodule \( P \) is a prime element in \( M \) if for any two submodules \( N, L \) of \( M \)

\[
N \ast L \subseteq P \Rightarrow N \subseteq P \text{ or } L \subseteq P.
\]

Those modules whose zero submodule is a prime element had been termed \(*\)-prime modules, i.e. \( N \ast L \neq 0 \) for all non-zero \( N, L \subseteq M \). Of course for \( M = R \), the \(*\)-product equals the product of left ideals and \( R \) is a \(*\)-prime \( R \)-module if and only if it is a prime ring. The meaning of the module theoretic prime concept for a ring \( R \) with Hopf algebra action \( H \) seen as left \( R \# H \)-module has been studied in [12] in connection with an open question in this area, due to Miriam Cohen, asking whether \( R \# H \) is a semiprime algebra provided \( R \) is semiprime and \( H \) is semisimple (see [5]).

The main purpose of this note is to shed new light into the following question which had been left open in [12]:

**Question.** Is the endomorphism ring of a \(*\)-prime module a prime ring ?

From [12, Proposition 4.2] it is known that the answer is yes, if the \(*\)-prime module \( M \) satisfies a light projectivity condition. Although we were unable to answer this question completely we will indicate various sufficient conditions for a \(*\)-prime module to have a prime endomorphism ring which narrows down the class of possible examples that could provide a negative answer.

Let \( M \) be a left \( R \)-module. \( S = \text{End}_R(M) \) shall always denote the endomorphism ring of \( M \). Since any \(*\)-prime module \( M \) has a prime annihilator ideal \( \text{Ann}(M) \) and since \( \text{Hom}_R(M, N) = \text{Hom}_{R/\text{Ann}(M)}(M, N) \) holds for any submodule \( N \) of \( M \), we will assume throughout this note that \( M \) is a faithful left module over a (unital, associative) prime ring \( R \).

### 1.1. Retractable modules

A \(*\)-prime module \( M \) is **retractable**, i.e. \( \text{Hom}(M, K) \neq 0 \) whenever \( 0 \neq K \subseteq M \). Note that it is always true that a retractable module with prime endomorphism ring is a \(*\)-prime module (see [12, Theorem 4.1]) and our question is whether this sufficient condition is also necessary. The retractability condition (called **quotient like** in [9] and **slightly compressible** in [14]) stems from the non-degeneration of the standard Morita context \((R, M, M^*, S)\) between a ring \( R \) and the endomorphism ring \( S \) of a module \( M \) via \( M^* = \text{Hom}(M, R) \) (see [17]). In the case of a group \( G \) acting on a ring \( R \), the
retractability of \( R \) as \( R \ast G \)-module says that every non-zero \( G \)-stable left ideal contains a non-zero fixed element. The Bergman-Isaacs theorem [2] says that \( R \) is retractable as left \( R \ast G \)-module if \( G \) is a finite group acting on a semiprime ring \( R \) such that no non-zero element of \( R \) has additive \(|G|\)-torsion. This fact had been used by Fisher and Montgomery in [8] to prove that \( R \ast G \) is semiprime provided \( R \) is semiprime and has no \(|G|\)-torsion, which originally with [6] motivated Cohen’s question for Hopf algebra actions.

For a locally nilpotent derivation \( \partial \) of a ring \( R \) it had been shown in [4, Lemma 3.8] that \( R \) is always retractable as \( R[x, \partial] \)-module. Rings \( R \) that are retractable as \( R \otimes R^{op} \)-module are those whose non-zero ideals contain non-zero elements like for example in the case of semiprime PI-rings ([13, Theorem 2]), central Azumaya rings ([15, 26.4]) or enveloping algebras of semisimple Lie algebras ([7, 4.2.2]). The retractability condition can be expressed by saying that the function from the lattice of left \( R \)-submodules of the module \( M \) to the lattice of left ideals of \( S \) defined as \( N \mapsto \text{Hom}(M, N) \) for submodules \( N \) of \( M \) has the property that the only submodule mapped to the zero left ideal of \( S \) is the zero submodule.

1.2. Endoprime modules

It is known by [12, 1.3] that the endomorphism ring of a right \( R \)-module \( M \) is prime if and only if \( \text{Hom}(M/N, M) = 0 \) for all non-zero fully invariant, \( M \)-generated submodules \( N \) of \( M \). With slightly different notation, Haghany and Vedadi defined a module \( M \) to be endoprime if \( \text{Hom}(M/K, M) = 0 \) for all non-zero fully invariant submodules \( K \) of \( M \) (see [9]). Thus endoprime modules have a prime endomorphism ring. Note that \( \text{Hom}(M, K)\text{Hom}(M/K, M) = 0 \) holds for all submodules \( K \) of \( M \). Hence a retractable module \( M \) with prime endomorphism ring \( S \) is endoprime. In other words a retractable module has a prime endomorphism ring if and only if it is endoprime. Since \( * \)-prime modules are retractable, our question can be equivalently reformulated to

**Question:** Are \( * \)-prime modules endoprime in the sense of Haghany and Vedadi?

1.3. Semi-projective modules

As mentioned before under a light projectivity condition our question has an affirmative answer. Recall from [15] that a module \( M \) is called semi-projective if any diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & K \\
\downarrow{g} & & \\
0 & & \\
\end{array}
\]

with \( K \subseteq M \) can be extended by some endomorphism of \( M \). In other words, \( M \) is semi-projective if and only if for any endomorphism \( f \) of \( M \) we have \( \text{Hom}(M, (M)f) = Sf \).
Lemma 1.1. ([12, Proposition 4.2]). A semi-projective module is \( \ast \)-prime if and only if it is a retractable module with prime endomorphism ring.

Let \( R \) be a ring and \( B \subseteq \text{End}_Z(R) \) be a subring of the ring of \( Z \)-linear endomorphisms of \( R \) such that all left multiplications \( L_a : R \to R \) defined by \( L_a(x) = ax \) for \( a, x \in R \) belong to \( B \). \( R \) becomes naturally a left \( B \)-module by evaluating of functions. The subring \( R_B = \{ (1) f \mid f \in B \} \) can be seen to be a generalized subring of invariants of \( R \) with respect to \( B \). It is not difficult to see, that \( R_B \) is isomorphic to \( \text{End}_B(R) \) (see [11, Lemma 1.8]). This general situation mimics the case of \( R \) considered as a bimodule or \( R \) considered having a Hopf algebra \( H \) acting on it. To ask that \( R \) is a semi-projective as \( B \)-module, is to say that for each \( x \in R_B \) one has \( R_B x = (Rx) \cap R_B \).

Considering \( R \) as a bimodule, we let \( B \) to be the subring of \( \text{End}_Z(R) \) generated by all left and right multiplications of elements of \( R \). The \( B \)-module structure of \( R \) is identical with the bimodule structure of \( R \). Then \( R \) is semi-projective as \( R \otimes R^{op} \)-module if for example all non-zero central elements of \( R \) are non-zero divisors in \( R \). Because if \( x \) is central and \( ax \) is central for some \( a \in R \), then for any \( b \in R \) one has \( (ab - ba)x = abx - bax = axb - axb = 0 \), i.e. \( ab = ba \) and \( a \) is central. Thus \( Rx \cap C(A) = C(A)x \). In case \( R \) is \( \ast \)-prime as \( R \otimes R^{op} \)-module, \( 0 \neq x \in C(R) \) and \( I = \text{Ann}(x) = \{ a \in R \mid ax = 0 \} \) is its annihilator, the \( \ast \)-product of \( I \) and \( Rx \) is given by:

\[
I \ast (Rx) = I\text{Hom}_{R \otimes R^{op}}(R, Rx) = I((Rx) \cap C(R)) \subseteq Ix = 0.
\]

Since we supposed that \( R \) is \( \ast \)-prime and \( x \neq 0 \), we get \( I = 0 \). This shows that no non-zero central element of \( R \) is a zero-divisor in \( R \). Consequently we can state the following

**Corollary 1.2.** A ring \( R \) is a \( \ast \)-prime \( R \otimes R^{op} \)-module if and only if the center of \( R \) is an integral domain and large in \( R \).

Here we say that a subring \( R' \) of \( R \) is large in \( R \) if any non-zero ideal of \( R \) contains a non-zero element of \( R' \).

Let \( G \) be a group acting on \( R \). It is known that \( R \) is a projective \( R \ast G \)-module if and only if \( G \) is a finite group and \( |G|1 \) is invertible in \( R \). Thus in this case \( R \) is a \( \ast \)-prime \( R \ast G \)-module if and only if \( R^G \) is a prime ring.

If \( R \) is an algebra over a field \( F \) and \( \partial \) is a locally nilpotent derivation of \( R \) and either \( \text{char}(F) = 0 \) or \( \partial \text{char}(F) = 0 \), then \( R \) is self-projective as left \( R[x, \partial] \)-module by [4, Proposition 3.10]. Hence in this situation (using also [4, Lemma 3.8]) \( R \) is a \( \ast \)-prime left \( R[x, \partial] \)-module if and only if \( R^G \) is a prime ring.
2. PRIME ENDOMORPHISM RINGS

The purpose of this section is to gather conditions for a \( \ast \)-prime module to have a prime endomorphism ring. Denote by \( \text{l.ann}_S(I) \) (resp. by \( \text{r.ann}_S(I) \)) the left (resp. right) annihilator in \( S \) of an ideal \( I \).

**Theorem 2.1.** The following statements are equivalent for a \( \ast \)-prime module \( M \) with endomorphism ring \( S \):

1. \( S \) is prime.
2. \( S \) is semiprime.
3. \( \text{l.ann}_S(I) \subseteq \text{r.ann}_S(I) \) holds for any ideal \( I \) of \( S \).
4. \( gSf = 0 \Rightarrow fSg = 0 \) for all \( f, g \in S \).

**Proof.**  
(a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) is trivial since the left and right annihilator of an ideal coincide in a semiprime ring.  
(c) \( \Rightarrow \) (a) Suppose that \( IJ = 0 \) for two ideals \( I, J \) of \( S \). Then \( \text{MHom}(M, MI)J \subseteq MIJ = 0 \) implies \( \text{Hom}(M, MI)J = 0 \). By (c) \( \text{JHom}(M, MI) = 0 \). Hence \( (MJ) \ast (MI) = MJ\text{Hom}(M, MI) = 0 \) and since \( M \) is \( \ast \)-prime, we have \( MI = 0 \) or \( MJ = 0 \), i.e. \( I = 0 \) or \( J = 0 \). Thus \( S \) is prime.

Condition (d) is equivalent to saying that
\[
\text{l.ann}_S(SfS) = \text{l.ann}_S(Sf) \subseteq \text{r.ann}_S(fS) = \text{r.ann}_S(SfS)
\]
for all \( f \in S \), which is a consequence of (c). On the other hand, assuming (d) condition (c) follows since for any non-zero ideal \( I \) we have \( \text{l.ann}_S(I) = \bigcap_{f \in I} \text{l.ann}_S(SfS) \) and the analogous statement for \( \text{r.ann}_S(I) \).

Note that (c) \( \Rightarrow \) (a) needed only the primeness condition for fully invariant sub-modules. These modules had been investigated by R. Wisbauer and I. Wijayanti and termed fully prime modules. We deduce two corollaries from the last theorem:

**Corollary 2.2.** Let \( M \) be a left \( R \)-module with endomorphism ring \( S \). Then \( S \) is prime and \( M \) is retractable if and only if \( M \) is \( \ast \)-prime and \( gSf = 0 \) implies \( fSg = 0 \) for all \( f, g \in S \).

As a particular case we recover the characterization of \( R \) being \( \ast \)-prime as bimodule (see 1.2):

**Corollary 2.3.** Let \( M \) be a left \( R \)-module with commutative endomorphism ring \( S \). Then \( M \) is \( \ast \)-prime if and only if \( M \) is retractable and \( S \) is an integral domain.

Since semiprime PI-rings or central Azumaya rings have large center, we see that any such ring is a \( \ast \)-prime bimodule if and only if its center is a domain. The next result generalizes the fact that semi-projective \( \ast \)-prime modules have prime endomorphism.
Proposition 2.4. Assume that for any non-zero ideal \( J \) of \( S \) which is essential as left and right ideal there exists a non-zero submodule \( N \) of \( M \) such that \( \text{Hom}(M, N) \subseteq J \). Then \( S \) is prime if \( M \) is \( \ast \)-prime.

Proof. Let \( I^2 = 0 \) for an ideal \( I \) of \( S \). Then \( J = r \cdot \text{ann}_S(I) \cap l \cdot \text{ann}_S(I) \) is a non-zero ideal of \( S \) which is essential on both sides. By assumption \( \text{Hom}(M, N) \subseteq J \) for some non-zero submodule \( N \) of \( M \). Thus \( MI \ast N = M \text{Hom}(M, N) \subseteq MIJ = 0 \) and as \( M \) is \( \ast \)-prime and \( N \) non-zero we have \( I = 0 \), i.e. \( S \) is semiprime and by Theorem 2.1 \( S \) is prime. 

A left \( R \)-module \( M \) is called torsionless if it is cogenerated by \( R \). A result by Amitsur says that any faithful torsionless module over a prime ring has a prime endomorphism ring (see [1, Corollary 2.8]). The following Proposition gives sufficient conditions for a \( \ast \)-prime module \( M \) to be torsionless.

Proposition 2.5. Let \( M \) be a faithful left \( R \)-module over a prime ring \( R \). In any of the following cases \( M \) is torsionless and hence has a prime endomorphism ring.

1. \( M \) is a \( \ast \)-prime module and is not a singular left \( R \)-module.
2. \( M \) is a \( \ast \)-prime module and \( R \) is a left duo ring, i.e. any left ideal is twosided.
3. \( M \) is non-singular and is cogenerated by all of its essential submodules.

Proof. Note that any non-zero submodule \( N \) of \( M \) that is not singular contains a submodule which is isomorphic to a non-zero left ideal of \( R \). To see this let \( 0 \neq x \in N \) be an element whose annihilator \( A = l \cdot \text{ann}_R(x) \) is not essential in \( R \). Let \( B \) be a complement of \( A \), i.e. a left ideal of \( R \) which is maximal with respect to \( A \cap B = 0 \). Then \( I = A \oplus B \) is an essential left ideal of \( R \) and \( Ix \neq 0 \) since \( B \) is non-zero. As \( B \simeq Ix \), we see that \( B \) is isomorphic to a submodule of \( M \).

1. As explained above, if \( M \) is not singular, then there exists a non-zero left ideal \( B \) of \( R \) which is isomorphic to a submodule of \( M \). Since \( M \) is cogenerated by any of its non-zero submodules, it is cogenerated by \( B \) and hence by \( R \) as \( B \subseteq R \). Thus \( M \) is torsionless.

2. Since \( M \) is a (faithful) prime module, every submodule is also faithful. By hypothesis \( I = 1 \cdot \text{ann}_R(m) \) is two sided for any element \( m \) of \( R \) and hence \( 0 = \text{Ann}(Rm) = \text{Ann}(R/I) = I \), i.e. \( M \) is not singular and the result follows from (1).

3. Let \( M \) be any non-zero nonsingular module that cogenerated by every essential submodule of itself. By Zorn’s Lemma there exists a maximal direct sum \( \bigoplus_i C_i \) of cyclic modules \( C_i = Rm_i \) non of which is singular. Let \( A_i = 1 \cdot \text{ann}_R(m_i) \) for each \( i \in I \). Since \( A_i \) is not essential in \( R \), there exists a non-zero complement \( B_i \) of \( A_i \) in \( R \) such that \( K_i = A_i \oplus B_i \) is an essential left ideal of \( R \). Let
Let $a$ be any element in $R$ such that $a \notin A_i$. Then there exists an essential left ideal $E$ of $R$ such that $Ea = Ra \cap K_i$. Because $M$ is nonsingular, we have $0 \neq Eam_i \subseteq K_im_i \cap Ram_i$. Thus $K_im_i$ is an essential submodule of $C_i$ and moreover $K_im_i \simeq B_i$. Hence $N = \bigoplus_{i \in I} K_im_i$ is essential in $M$ and by hypothesis cogenerates $M$. Since $N \simeq \bigoplus_{i \in I} B_i \subseteq R^{(I)}$, $M$ is torsionless.

The Wisbauer category of a module $M$ is the full subcategory of $R$-Mod consisting of submodules of quotients of direct sums of copies of $M$. For $M = R$, we have $\sigma[R] = R$-Mod. A module $N \in \sigma[M]$ is called $M$-singular if there are modules $K, L \in \sigma[M]$ with $K$ being an essential submodule of $L$ and $N \simeq L/K$. For $M = R$, $R$-singular modules are called singular. A module $M$ is called polyform or non-$M$-singular if it does not contain any $M$-singular submodule or equivalently if $\text{Hom}(L/K, M) = 0$ for all essential submodules $K \subseteq L \subseteq M$ (see [15]).

**Proposition 2.6.** The endomorphism ring of a $*$-prime polyform module is a prime ring.

**Proof.** Recall our general hypothesis that $M$ is a faithful left module over a prime ring $R$. Let $I^2 = 0$ for some ideal $I$ of $S$. Then $MI$ is fully invariant. Note that any fully invariant submodule $N$ of $M$ is essential as $M$ is $*$-prime, because for any non-zero submodule $L$ of $M$ we have that $0 \neq N \ast L = N\text{Hom}(M, L) \subseteq N \cap L$. Thus $MI$ is essential in $M$. Denote by $\pi : M \rightarrow M/MI$ the canonical projection, then $\pi I \subseteq \text{Hom}(M/MI, M) = 0$ as $M$ is polyform. Thus $I = 0$ and $S$ is semiprime. By Theorem 2.1 $S$ is prime.

3. **Simple Submodules in Weakly Compressible Modules**

The purpose of this section is to see what can be said about the endomorphism ring of a $*$-prime module with non-zero socle. It is also clear that if a $*$-prime module contains a simple submodule $S$, then any simple submodule of $M$ must be isomorphic to $S$. Moreover since $\text{Soc}(M)$, the socle of $M$, is fully invariant, we have for any submodule $L$ of $M$: $\text{Soc}(M) \ast L \subseteq \text{Soc}(M) \cap L$. Thus a $*$-prime module $M$ has either zero socle or has an essential and homogeneous semisimple socle, i.e. isomorphic to a direct sum of copies of a simple module.

A submodule $N$ is called semiprime if for any $K \subseteq M : K \ast K \subseteq N \Rightarrow K \subseteq N$. A module whose zero submodule is semiprime is called weakly compressible by Zelmanowitz (see [16]). Obviously $*$-prime modules are weakly compressible.

**Lemma 3.1.** Any simple submodule of a weakly compressible module $M$ is a direct summand.

**Proof.** Let $K$ be a simple submodule of $M$, then $0 \neq K \ast K = KHom(M, K)$ implies the existence of $f : M \rightarrow K$ such that $f(K)$ is non-zero, i.e. $f(K) = K$ as
$K$ is simple. By Schur’s Lemma $\text{End}(K)$ is a division ring and hence there exists an inverse $g \in \text{End}(K)$ of $f$ restricted to $K$, i.e. $gf = id_K$. Considering $g$ as a map from $K$ to $M$ we showed that $f$ splits, i.e. $K$ is a direct summand of $M$.

Since by the last Lemma, simple modules of a $\ast$-prime module are direct summands, we have the following

**Corollary 3.2.** Any weakly compressible module with DCC or ACC on direct summands and non-zero socle is homogeneous semisimple.

Recall that a ring $R$ is said to be left quotient finite dimensional (qfd) if every cyclic left $R$-module has finite Goldie dimension. Any left noetherian or more general any ring with Krull dimension is qfd.

**Theorem 3.3.** Let $R$ be a semilocal or a left qfd ring, then any $\ast$-prime module with non-zero socle has a prime endomorphism ring.

**Proof.** If $M$ is a $\ast$-prime module with a non-zero socle, then $\text{Soc}(M)$ is essential and homogeneous semisimple. Any cyclic $C$ submodule of $M$ is also a $\ast$-prime module with non-zero essential socle and by assumption has finite Goldie dimension (in case $R$ is qfd) or finite dual Goldie dimension (in case $R$ is semilocal). In either case $C$ has ACC on direct summands and by Corollary 3.2 $C$ is homogeneous simple. Thus $M = \text{Soc}(M) \cong E^{(\Lambda)}$ is homogeneous semisimple and $\text{End}(M) \cong \text{End}(E^{(\Lambda)})$ is a prime ring.

Recall that a ring $R$ has left Krull dimension 0 if it is left artinian and left Krull dimension 1 if every proper cyclic left $R$-module $M \neq R$ is artinian.

**Proposition 3.4.** Any $\ast$-prime left module over a ring with left Krull dimension less or equal to 1 has a prime endomorphism ring.

**Proof.** Let $R$ be a ring with left Krull dimension $\leq 1$ and let $M$ be a $\ast$-prime left $R$-module. If $M$ is not singular, then it has a prime endomorphism ring by Proposition 2.5. Suppose that $M$ is singular and let $C$ be a non-zero cyclic submodule of $M$, then $C$ is also singular and hence proper, i.e. $C \simeq R/I$ with $I \neq 0$. By hypothesis $R$ has Krull dimension $\leq 1$ and thus $C$ is artinian. This shows that $M$ has a non-zero socle. By 3.3 $M$ has a prime endomorphism.

This implies that for instance any $\ast$-prime module over the first Weyl algebra $A_1$ has a prime endomorphism ring.

4. Conclusion

Let $C(R)$ denote the center of $R$. Faithful $\ast$-prime modules $M$ that are not singular have prime endomorphism ring by Proposition 2.5. This applies in particular to the following case:
(1) if $M$ has a non-zero submodule which is finitely generated over $C(R)$ or
(2) if $R$ has a non-zero left ideal which is finitely generated over $C(R)$ or
(3) if $R$ has a non-zero left socle.

In case (1), if $N = C(R)x_1 + \cdots + C(R)x_n$, then

$$0 = \text{Ann}(M) = \text{Ann}(N) = \text{Ann}(x_1) \cap \cdots \cap \text{Ann}(x_n).$$

Thus not all of the elements $x_i$ can be singular and $M$ is not a singular module. Case (2) reduces to the first case, because if $I$ is a non-zero left ideal of $R$ which is finitely generated over $C(R)$, then since $M$ is faithful, there must exist a non-zero element $m \in M$ with $N = Im \neq 0$. But then $N$ is a non-zero submodule of $M$ which is finitely generated over $C(R)$ and (1) applies.

In case (3) we also see that due to $0 = \text{Ann}(M) = \bigcap_{x \in M} \text{Ann}(x)$ not all the annihilators $\text{Ann}(x)$ can be essential left ideals, since otherwise the left socle would be contained in $\text{Ann}(M)$ and would be zero. Hence $M$ is not a singular module.

From the preceding we can conclude that if there exists a $*$-prime faithful left $R$-module $M$ whose endomorphism ring is not prime, then

- $R$ is not a left duo ring;
- $R$ has zero left socle
- $R$ does not contain any non-zero left ideal which is finitely generated over $C(R)$;
- the Krull dimension of $R$ is greater than 1;
- $\text{End}(M)$ is not commutative;
- $M$ is a singular left $R$-module which is neither torsionless nor semi-projective;
- $M$ is not polyform, i.e. $M$ is cogenerated by some $M$-singular submodule;
- no non-zero submodule of $M$ is finitely generated over the center $C(R)$ of $R$;
- if $M$ has non-zero socle, then $R$ cannot be semilocal nor can $R$ have Krull dimension.

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