AN INVERSE NODAL PROBLEM AND AMBARZUMYAN PROBLEM FOR THE PERIODIC $p$-LAPLACIAN OPERATOR WITH INTEGRABLE POTENTIALS

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Abstract. In this note, we solve the inverse nodal problem and Ambarzumyan problem for the $p$-Laplacian coupled with periodic or anti-periodic boundary conditions. We also extend some results in a previous paper to $p$-Laplacian with $L^1$ potentials, and for arbitrary linear separated boundary conditions. There we prove a generalized Riemann-Lebesgue Lemma which is of independent interest.

1. INTRODUCTION

An inverse nodal problem is a problem of understanding the potential function through the nodal points of eigenfunctions, without any other spectral information. An Ambarzumyan problem is the unique determination of potential $q$, when its associated spectrum $\sigma(q) = \sigma(0)$. Both problems have been well studied for the classical Sturm-Liouville operator (see \[8, 9, 11, 14\]). In a previous paper, we studied the $p$-Laplacian operator with $C^1$-potentials and solved the inverse nodal problem and Ambarzumyan problem for Dirichlet boundary conditions \[10\]. Now we want to extend the results to periodic/anti-periodic boundary conditions, and to $L^1$ potentials, which is the most general class of potentials.

Consider the equation

\[
-\left(\left(y^{(p-1)}\right)\right)' = (p-1)(\lambda - q(x))y^{(p-1)},
\]

where $f^{(p-1)} = |f|^{p-1}\text{sgn}f$. Assume that $q(1 + x) = q(x)$ for $x \in \mathbb{R}$, then (1.1) can be coupled with periodic or anti-periodic boundary conditions respectively:

\[
y(0) = y(1), \quad y'(0) = y'(1)
\]

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or

\begin{equation}
(1.3) \quad y(0) = -y(1), \quad y'(0) = -y'(1).
\end{equation}

When \( p = 2 \), the above is the classical Hill’s equation. It follows from Floquet theory that there are countably many interlacing pairs of periodic and anti-periodic eigenvalues of Hill’s operator. However, Floquet theory does not apply when \( p \neq 2 \). Let \( \sigma_{2k} \) (resp. \( \sigma_{2k-1} \)) denote the set of periodic (resp. anti-periodic) eigenvalues of (1.1) which admit eigenfunctions with exactly \( 2k \) (resp. \( 2k-1 \)) zeros in \([0, 1)\). In 2001, Zhang [15] used a rotation number function to show the existence of the minimal eigenvalue \( \Delta_n = \min \sigma_n \) and the maximal eigenvalue \( \lambda_n = \max \sigma_n \) respectively. Binding and Rynne studied in more detail in a series of papers [3, 4, 5] and showed that

(i) \( \sigma_{2k} \) and \( \sigma_{2k-1} \) are nonempty and compact. Also for all \( \lambda \in \sigma_{2k} \),

\[ \lambda_{2k-1} < \delta_{2k} \leq \lambda \leq \lambda_{2k} < \lambda_{2k+1}, \]

while \( \sigma_0 = \{ \lambda_0 \} \) contains only one simple eigenvalue.

(ii) There exists a sequence of variational periodic eigenvalues \( \{ \gamma_n \} \) and variational anti-periodic eigenvalues \( \{ \delta_n \} \), such that \( \gamma_0 = \lambda_0 \) and for all \( k \geq 1 \),

\[ \lambda_{2k} = \gamma_{2k} \geq \delta_{2k} = \gamma_{2k-1} > \lambda_{2k-1} = \delta_{2k-1}. \]

Furthermore, letting \( \mu_n \) (\( n \geq 1 \)) and \( \nu_n \) (\( n \geq 0 \)) be the Dirichlet and Neumann eigenvalues which admit eigenfunctions with exactly \( n \) zeros in \([0, 1)\), we have

\[ \lambda_{2k} \geq \mu_{2k}, \nu_{2k} \geq \lambda_{2k}, \]

\[ \lambda_{2k-1} \geq \mu_{2k-1}, \nu_{2k-1} \geq \lambda_{2k-1}. \]

The variational periodic eigenvalues \( \{ \gamma_n \} \) are defined by the Ljusternik-Schnirelmann construction. Define

\[ W^1_p(0, 1) = \{ w \in W^1_p(0, 1) : w(0) = w(1), \ w'(0) = w'(1) \}. \]

Let \( M = \{ u \in W^1_p(0, 1) : \int_0^1 |u|^p = 1 \}, \) and

\[ A = \{ A \subset M : \ A \text{ is non-empy, compact and symmetric } (A = -A) \}. \]

Hence we define the Krasnoselskij genus of \( A \in A \) by

\[ \varphi(A) = \min \{ m \in \mathbb{N} : \ \text{there exists a continuous, odd } f : A \rightarrow \mathbb{R}^m \setminus \{0\} \}. \]

Thus for any integer \( n \geq 0 \), let \( \mathcal{F}_n = \{ A \in A : \ \varphi(A) \geq n \}. \) Then

\[ \gamma_n := \min_{A \in \mathcal{F}_{n+1}} \max_{u \in A} \int_0^1 \left( \frac{|u|^p}{p-1} + q|u|^p \right). \]

The set of variational anti-periodic eigenvalues \( \{ \delta_n \} \) is defined in a similar manner.
(iii) In general, non-variational eigenvalues may exist in $\sigma_{2k}$ and $\sigma_{2k-1}$ for all $k \geq 1$.

Some of the above properties are similar to the linear case, but others are not. This makes the study of $p$-Laplacian operators more interesting.

From now onward, by a periodic eigenvalue $\lambda_{2k}$, we mean an element of $\sigma_{2k}$, whether it is variational or non-variational or not. By an anti-periodic eigenvalue $\lambda_{2k-1}$, we mean an element of $\sigma_{2k-1}$, variational or non-variational.

In 2008, Brown and Eastham [6] derived a sharp asymptotic expansion of periodic eigenvalues of the $p$-Laplacian with locally integrable and absolutely continuous $(r-1)$ derivative potentials respectively. Below is a version of their theorem for periodic eigenvalues of the $p$-Laplacian (1.1), (1.2).

**Theorem 1.1.** ([6, Theorem 3.1]). Let $q$ be 1-periodic and locally integrable in $(-\infty, \infty)$. Then the periodic eigenvalue $\lambda_{2k}$ satisfies

$$
\lambda_{2k}^{1/p} = 2k\hat{\pi} + \frac{1}{p(2k\hat{\pi})^{p-1}} \int_0^1 q(t)dt + o\left(\frac{1}{k^{p-1}}\right),
$$

where $\hat{\pi} = \frac{2\pi}{p \sin(\frac{\pi}{p})}$.

By a similar argument, the asymptotic expansion of the anti-periodic eigenvalue $\lambda_{2n-1}$ satisfies

$$
\lambda_{2n-1}^{1/p} = (2k - 1)\hat{\pi} + \frac{1}{p((2k - 1)\hat{\pi})^{p-1}} \int_0^1 q(t)dt + o\left(\frac{1}{k^{p-1}}\right).
$$

We denote by $\{x_{i}^{(n)}\}_{i=0}^{n-1}$ the zeros of the eigenfunction corresponding to a periodic/anti-periodic eigenvalue $\lambda_{n}$, and define the nodal length $\ell_{i}^{(n)} = x_{i+1}^{(n)} - x_{i}^{(n)}$ and $j = j_{n}(x) = \max\{i : x_{i}^{(n)} \leq x\}$. Our main theorem is as follows.

**Theorem 1.2.** Let $q \in L^1(0,1)$ be 1-periodic. Define $F_{n}(x)$ as the following:

(a) For the periodic case, let

$$
F_{2k}(x) = p((2k\hat{\pi})^p[(2k\hat{\pi})^p(2k) - 1] + \int_0^1 q(t)dt,
$$

(b) For the anti-periodic case, let

$$
F_{2k-1}(x) = p((2k - 1)\hat{\pi})^p[(2k - 1)\hat{\pi})^p(2k - 1) - 1] + \int_0^1 q(t)dt.
$$

Then both $\{F_{2k}\}$ and $\{F_{2k-1}\}$ converges to $q$ pointwise a.e. and in $L^1(0,1)$.

Thus either one of the sequences $\{F_{2k}\}/\{F_{2k-1}\}$ will be sufficient to reconstruct $q$. Note that here $q \in L^1(0,1)$. Furthermore, the map between the nodal space and the set of admissible potentials are homeomorphic after a partition (cf. [10]). The same idea also works for linear separated boundary value problems with integrable potentials.
Using the eigenvalue asymptotics above, the Ambarzumyan problems for the periodic and anti-periodic boundary conditions can also be solved.

**Theorem 1.3.** Let \( q \in L^1((0, 1)) \) be periodic of period 1.
(a) If a sequence of periodic eigenvalues \( \{\lambda_{2k}\}_{k=0}^\infty \) for (1.1) such that \( \lambda_{2k} \in \sigma_2 \), is given by \( \lambda_{2k} = (2k\pi)^p \) for all \( k \in \mathbb{N} \cup \{0\} \), then \( q = 0 \) on \([0, 1]\).
(b) If a set of anti-periodic eigenvalue \( \{\lambda_{2k-1}\}_{k=1}^\infty \) for (1.1) such that \( \lambda_{2k-1} \in \sigma_{2k-1} \), is given by \( \lambda_{2k-1} = ((2k-1)\pi)^p \) for all \( k \in \mathbb{N} \), with \( \lambda_1 = \min \sigma_1 \), and \( \int_0^1 q(t)S_p(\pi t)^p \, dt = 0 \), then \( q = 0 \) on \([0, 1]\).

Note that this sequence might not exploit all the periodic eigenvalues, as we know that the set \( \sigma_{2k} \) \( (k \geq 1) \) contains at least two variational periodic eigenvalues \( (\Lambda_{2k} \text{ and } \Lambda_{2k}) \), as well as some non-variational periodic eigenvalues, as explained above. In fact, it has been shown that when \( p \neq 2 \), the set \( \sigma_{2k} \) can have arbitrarily many elements for \( C^1 \) potentials (cf. [3, Theorem 1.3]). The situation for anti-periodic eigenvalues is similar.

In Section 2, we shall apply Theorem 1.1 to study the problems involving periodic and anti-periodic boundary conditions. There Theorem 1.1 and Theorem 1.2 will be proved. In section 3, we shall deal with the case of linear separated boundary conditions.

Recently, we worked on a Tikhonov regularization approach of the inverse nodal problem for \( p \)-Laplacian [7]. The approach helps to obtain a more practical approximation of the potential function for Dirichlet \( p \)-Laplacian eigenvalue problem. The present work will be useful in making a similar approach for the periodic \( p \)-Laplacian eigenvalue problem.

### 2. PROOF OF MAIN RESULTS

Fix \( p > 1 \) and assume that \( q = 0 \) and \( \lambda = 1 \). Then (1.1) becomes

\[-(y^{(p-1)})' = (p-1)y^{(p-1)}.\]

Let \( S_p \) be the solution satisfying the initial conditions \( S_p(0) = 0, S_p'(0) = 1 \). It is well known that \( S_p \) and its derivative \( S_p' \) are periodic functions on \( \mathbb{R} \) with period \( 2\pi \). The two functions also satisfy the following identities (cf. [6, 10]).

**Lemma 2.1.**

(a) \( |S_p(x)|^p + |S_p'(x)|^p = 1 \) for any \( x \in \mathbb{R} \);
(b) \( (S_pS_p^{(p-1)})' = |S_p|^p - (p-1)|S_p|^p = 1 - p|S_p|^p = (1-p) + p|S_p|^p \).

Next we define a generalized Prüfer substitution using \( S_p \) and \( S_p' \):

\[
(2.1) \quad y(x) = r(x)S_p(\lambda^{1/p}\theta(x)), \quad y'(x) = \lambda^{1/p}r(x)S_p'(\lambda^{1/p}\theta(x)).
\]

By Lemma 2.1, one obtains ([10])

\[
(2.2) \quad \theta'(x) = 1 - \frac{q(x)}{\lambda}|S_p(\lambda^{1/p}\theta(x))|^p.
\]
Theorem 2.2. In the periodic/anti-periodic eigenvalue problem, if \( q \in L^1(0, 1) \) is periodic of period 1, then

\[
q(x) = \lim_{n \to \infty} p\lambda_n \left( \frac{\lambda_n^{1/p} \ell_j(n)}{\pi} - 1 \right),
\]

pointwise a.e. and in \( L^1(0, 1) \), where \( j = j_n(x) = \max \{ k : x_k^{(n)} \leq x \} \).

The proof below works for both even and odd \( n \)’s, i.e. for both periodic and anti-periodic problems. Some of the arguments above are motivated by [9]. See also [11].

Proof. First, integrating (2.2) from \( x_k^{(n)} \) to \( x_{k+1}^{(n)} \) with \( \lambda = \lambda_n \), we have

\[
\frac{\pi}{\lambda_n^{1/p}} = \ell_k^{(n)} - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt = \ell_k^{(n)} - \frac{1}{\lambda_n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt - \frac{1}{\lambda_n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p} \theta(t))|^{p} - \frac{1}{p}) dt.
\]

Hence,

\[
(2.3) \quad \ell_k^{(n)} = \frac{\pi}{\lambda_n^{1/p}} + \frac{1}{p\lambda_n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt + \frac{1}{\lambda_n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p} \theta(t))|^{p} - \frac{1}{p}) dt.
\]

and

\[
(2.4) \quad \frac{p\lambda_n}{\pi} \left( \frac{\lambda_n^{1/p} \ell_j(n)}{\pi} - 1 \right) = \frac{\pi}{\lambda_n^{1/p}} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt + \frac{p\lambda_n^{1/p}}{\pi} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t)(|S_p(\lambda_n^{1/p} \theta(t))|^{p} - \frac{1}{p}) dt.
\]

Now, for \( x \in (0, 1) \), let \( j = j_n(x) = \max \{ k : x_k^{(n)} \leq x \} \). Then \( x \in I_j^{(n)} := [x_j^{(n)}, x_{j+1}^{(n)}] \) and, for large \( n \),

\[
I_j^{(n)} \subset B(x, \frac{2\pi}{\lambda_n^{1/p}}),
\]

where \( B(t, \varepsilon) \) is the open ball with centre \( t \) and radius \( \varepsilon \). That is, the sequence of intervals \( \{ I_j^{(n)} : n \text{ is sufficiently large} \} \) shrinks to \( x \) nicely (cf. Rudin [13, p.140]).

Since \( q \in L^1(0, 1) \) and \( \frac{\lambda_n^{1/p} \ell_j(n)}{\pi} = 1 + o(1) \), we define the sequence of functions

\[
h_n := \frac{\lambda_n^{1/p}}{\pi} \sum_{k=0}^{n-1} \left( \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q \right) \chi_{I_k^{(n)}}.
\]
which is convergent to $q$ pointwise a.e. $x \in (0, 1)$. Furthermore,

$$|h_n| \leq g_n := \frac{\lambda_n^{1/p}}{\pi} \sum_{k=0}^{n-1} \left( \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} |q| \right) x_{k}^{(n)},$$

and as $n$ tends to infinity,

$$\int_{0}^{1} g_n(t) dt = \sum_{k=0}^{n-1} \frac{\lambda_n^{1/p} x_{k}^{(n)}}{\pi} \left| \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt \right| \to \|q\|_1 .$$

Thus when $n$ is large, $|h_n - q| \leq (2g_n + |q|)$ and the integral of the latter converges to $3\|q\|_1$. By the general Lebesgue dominated convergence theorem [12, p.89], $h_n$ converges to $q$ in $L^1(0, 1)$.

On the other hand, let $q_{k,n} := \frac{1}{h_k^n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} q(t) dt$. Then $\sum_{k=0}^{n-1} q_{k,n} f_{\lambda_k^{(n)}}$ converges to $q$ pointwise a.e. Let $\phi_n(t) = |S_p(\lambda_n^{1/p} \theta(t))|^p - \frac{1}{p}$. Then for a.e. $x \in (0, 1),

$$T_n(x) := \frac{p\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q(t) \phi_n(t) dt ,$$

$$= \frac{p\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} (q(t) - q_{j,n}) \phi_n(t) dt + \frac{p\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} q_{j,n} \phi_n(t) dt ,$$

$$= A_n(x) + B_n(x) .$$

By Lemma 2.1(b) and (2.2),

$$B_n(x) = \frac{p\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} \left( |S_p(\lambda_n^{1/p} \theta(t))|^p - \frac{1}{p} \right) \left( \theta'(t) + \frac{q(t)}{\lambda_n^{1/p}} |S_p(\lambda_n^{1/p} \theta(t))|^p \right) dt ,$$

$$= - \frac{pq_j n}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} S_p(\lambda_n^{1/p} \theta(t)) S_p' (\lambda_n^{1/p} \theta(t)) (p-1) |S_p(\lambda_n^{1/p} \theta(t))|^p dt + O(\lambda_n^{1+1/p}) ,$$

$$= O(\lambda_n^{-1+1/p}) .$$

Also,

$$|A_n(x)| \leq \frac{p\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}| |S_p(\lambda_n^{1/p} \theta(t))|^p - \frac{1}{p} |dt| ,$$

$$\leq \frac{(p - 1)\lambda_n^{1/p}}{\pi} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} |q(t) - q_{j,n}| dt ,$$
which converges to 0 pointwise a.e. because the sequence of intervals \( \{ I_j^{(n)} : n \text{ is sufficiently large} \} \) shrinks to \( x \) nicely. We conclude that \( T_n \to 0 \) a.e. \( x \in (0, 1) \).

Finally, applying the general Lebesgue dominated convergence theorem as above, \( T_n \to 0 \) in \( L^1(0, 1) \). Therefore, the left hand side of (2.4) converges to \( q \) pointwise a.e. and in \( L^1(0, 1) \).

**Proof of Theorem 1.2.** By the eigenvalue estimates (1.4) and (1.5), we have

\[
(2.5) \quad p \lambda_{2k} \left( \frac{\lambda_0^{1/p} \ell^{(2k)}_{2k}(x)}{\ell^{(2k)}_{j_k}(x)} - 1 \right) = p(2k)^p \left( 2k - 1 \right) + 2k \int_0^1 q(t) dt + o(1) .
\]

Hence by Theorem 2.2 and the fact that \( 2k \ell^{(2k)}_{j_k} = 1 + o(1) \),

\[
F_{2k}(x) = p(2k)^p \left( 2k - 1 \right) + \int_0^1 q(t) dt
\]

also converges to \( q \) pointwise a.e. and in \( L^1(0, 1) \). The proof for (b) is the same.

**Proof of Theorem 1.3.** By (1.4), we have \( \int_0^1 q(t) dt = 0 \). Also as the least periodic eigenvalue \( \lambda_0 = 0 \) is variational, we take the constant function 1 as a test function. Then

\[
0 = \lambda_0 \leq \int_0^1 q = 0.
\]

Therefore 1 is the first periodic eigenfunction, and \( q = 0 \). This proves (a).

For part (b), since \( \lambda_{2k-1} = ((2k-1)\pi)^p \) for \( k \in \mathbb{N} \), we have, by (1.5), \( \int_0^1 q(t) dt = 0 \). Moreover, \( v(x) = p^{1/p} S_p(\pi x) \) satisfies anti-periodic boundary conditions and \( \|v\|_{L^p} = 1 \). Note that by Lemma 2.1(b),

\[
\int_0^1 |S_p(\pi t)|^p dt - \frac{p-1}{p} = \int_0^1 |S_p(\pi t)|^p dt - \frac{1}{p} = 0 .
\]

Now \( \lambda_1 = \pi^p \) is the first minimal anti-periodic eigenvalue, so it is a variational one. We let \( v \) be a test function, and obtain by variational principle and the hypothesis, that

\[
\pi^p \leq \int_0^1 \frac{p^{p-1}}{p} |S_p(\pi t)|^p dt + p \int_0^1 q(t) S_p(\pi t)^p dt = \pi^p .
\]

This implies \( v \) is the first eigenfunction. Thus \( q = 0 \) a.e. in \((0, 1)\).

**3. LINEAR SEPARATED BOUNDARY CONDITIONS**

Consider the one-dimensional \( p \)-Laplacian with linear separated boundary conditions...
\begin{align}
  \begin{cases}
    y(0)S_p'(\alpha) + y'(0)S_p(\alpha) = 0 \\
    y(1)S_p'(\beta) + y'(1)S_p(\beta) = 0
  \end{cases},
\end{align}

where \( \alpha, \beta \in [0, \pi) \). Letting \( \mu_n \) be the \( n \)th eigenvalue whose associated eigenfunction has exactly \( n - 1 \) zeros in \((0, 1)\), the generalized phase \( \theta_n \) as given in (2.2) satisfies

\[ \theta_n(0) = \frac{-1}{\mu_n^{1/p}} \tilde{C}_T^{-1}(\alpha); \]

\[ \theta_n(1) = \frac{1}{\mu_n^{1/p}} \left( n\pi - \tilde{C}_T^{-1}(\beta) \right), \]

where the function \( \tilde{C}_T(\gamma) := \frac{S_p(\gamma)}{S_p'(\gamma)} \) is an analogue of cotangent function, while \( \tilde{C}_T(\gamma) := CT_p(\gamma) \) if \( \gamma \neq 0 \); and \( CT_p(0) := 0 \). Also \( \tilde{C}_T^{-1} \) stands for the inverse of \( \tilde{C}_T \), taking values only in \([0, \pi)\).

Let \( \phi_n(x) = |S_p(\mu_n^{1/p}\theta_n(x))| - \frac{1}{\mu_n} \). Below we shall state a general Riemann-Lebesgue lemma, which shows that \( \int_0^1 g\phi_n \to 0 \) for any \( g \in L^1(0, 1) \), when \( \mu_n \)'s are associated with certain linear separated boundary conditions. In the case of periodic boundary conditions, Brown and Eastham [6] used a Fourier series expansion of \( \phi_n \) where \( \phi_n(\mu_n^{1/p}\theta_n(x)) \approx \phi_n(\alpha + 2n\pi x) \) and apply Plancherel Theorem to show convergence.

**Lemma 3.1.** Let \( f_n \) be uniformly bounded and integrable on \((0, 1)\). Suppose that

(i) there exists a partition \( \{x_0^n = 0 < x_1^n < \cdots < x_n^n = 1\} \) such that \( \Delta x_k^n := x_{k+1}^n - x_k^n = o(1) \) as \( n \to \infty \);

(ii) \( F_k^n(x) := \int_{x_k^n}^x f_n(t) \, dt \) satisfies \( F_k^n(x) = O\left( \frac{1}{n} \right) \) for \( x \in (x_k^n, x_{k+1}^n) \) and \( F_k^n(x_{k+1}^n) = o\left( \frac{1}{n} \right) \) for all \( 0 \leq k \leq n - 1 \), as \( n \to \infty \).

Then for any \( g \in L^1(0, 1) \), \( \int_0^1 g\phi_n \to 0 \) as \( n \to \infty \).

**Proof.** Let \( |f_n| \leq M \). We divide the proof into two parts. First, suppose that \( g \in C^1[0, 1] \). We can find a constant \( M_1 > 0 \) such that \( |g|, |g'| \leq M_1 \). Given any \( \epsilon > 0 \), then for sufficiently large \( n \), we have \( \Delta x_k^n \leq \epsilon \), and \( |F_k^n(x_{k+1}^n)| \leq \frac{\epsilon}{2M_1n} \), \( |F_k^n(x)| \leq \frac{\epsilon}{2M_1n} \) for \( x \in (x_k^n, x_{k+1}^n) \) for all \( 0 \leq k \leq n - 1 \). Using integration by parts,

\[ \left| \int_0^1 g\phi_n \right| = \left| \sum_{k=0}^{n-1} \int_{x_k^n}^{x_{k+1}^n} g\phi_n \right| = \left| \sum_{k=0}^{n-1} \left( g(x_{k+1}^n)F_k^n(x_{k+1}^n) - \int_{x_k^n}^{x_{k+1}^n} g'F_k^n \right) \right| \leq \epsilon. \]
Take any \( g \in L^1(0, 1) \). Then there is a \( C^1 \) function \( \tilde{g} \) on \([0, 1]\) such that \( \int_0^1 |\tilde{g} - g| < \epsilon \). Hence
\[
\int_0^1 g f_n = \int_0^1 (g - \tilde{g}) f_n + \int_0^1 \tilde{g} f_n.
\]
Here \( |\int_0^1 (g - \tilde{g}) f_n| \leq M \epsilon \), and by above, the term \( \int_0^1 \tilde{g} f_n \) can be arbitrarily small when \( n \) is large enough. Hence the theorem is valid. \( \blacksquare \)

**Corollary 3.2.** Consider the \( p \)-Laplacian (1.1) with boundary conditions (3.1). Define \( \phi_n(x) = |S_p(\mu_n^{1/p} \theta_n(x))|^{p - 1} - \frac{1}{p} \), then for any \( g \in L^1(0, 1) \), \( \int_0^1 g \phi_n \to 0 \).

**Proof.** Since \( \theta_n(0) \) and \( \theta_n(1) \) are as given in (3.2), \( \phi_n \) is uniformly bounded on \([0, 1]\). Take \( x_k^n \) be such that \( \theta(x_k^n) = \frac{k \pi}{\mu_n} \). Also by integrating the phase equation (2.2), \( \mu_n^{1/p} = O(n) \), and
\[
\Delta x_n = O\left(\frac{1}{\mu_n^{1/p}}\right) = O\left(\frac{1}{n}\right).
\]
Hence by Lemma 2.1(b) and (3.1), we have for \( k = 1, \ldots, n - 2 \),
\[
\int_{x_k^n}^{x_{k+1}^n} \phi_n(x) \, dx = \frac{-1}{p \mu_n^{1/p}} \int_{x_k^n}^{x_{k+1}^n} \frac{1}{\theta_n'(x)} \, dx \left[ S_p(\mu_n^{1/p} \theta_n(x)) S_p'(\mu_n^{1/p} \theta_n(x)) (p-1) \right] \, dx
\]
\[
= \frac{-1}{p \mu_n^{1/p}} \left[ S_p(\mu_n^{1/p} \theta_n(x)) S_p'(\mu_n^{1/p} \theta_n(x)) (p-1) \right]_{x_k^n}^{x_{k+1}^n} + O\left(\frac{1}{\mu_n}\right)
\]
\[
= O\left(\frac{1}{\mu_n^{1/p}}\right) = o\left(\frac{1}{n}\right),
\]
since \( S_p(k \pi) = 0 \). It is also clear that \( \int_{x_k^n}^{x} \phi_n(x) \, dx = O\left(\frac{1}{n}\right) \). Thus we may apply Lemma 3.1 to complete the proof. \( \blacksquare \)

**Theorem 3.3.** When \( q \in L^1(0, 1) \), the eigenvalues \( \mu_n \) of the Dirichlet \( p \)-Laplacian (1.1) satisfies, as \( n \to \infty \),
\[
\mu_n^{1/p} = n \pi + \frac{1}{p(n \pi)^{p-1}} \int_0^1 q(t) \, dt + o\left(\frac{1}{n^{p-1}}\right).
\]
Furthermore, \( F_n \) converges to \( q \) pointwise and in \( L^1(0, 1) \), where
\[
F_n(x) := p(n \pi)^{p} (n \ell_j^{(n)} - 1) + \int_0^1 q(t) \, dt.
\]
Proof. Integrating (2.2) from 0 to 1, we have
\[
\mu_n^{1/p} = n\hat{\pi} + \frac{1}{p\mu_n^{1-1/p}} \int_0^1 q(t)|S_p(\mu_n^{1/p}\theta(t))|^p dt,
\]
\[
= n\hat{\pi} + \frac{1}{p\mu_n^{1-1/p}} \int_0^1 q(t)dt + \frac{1}{p\mu_n^{1-1/p}} \int_0^1 q(t)(|S_p(\mu_n^{1/p}\theta(t))|^p - \frac{1}{p}) dt.
\]
Then by Corollary 3.2, we have
\[
\int_0^1 q(t)(|S_p(\mu_n^{1/p}\theta(t))|^p - \frac{1}{p}) dt = o(1),
\]
for any \( q \in L^1(0, 1) \). Hence (3.3) holds. Furthermore, by Theorem 2.2, we can obtain the reconstruction formula with pointwise and \( L^1 \) convergence. 

Remark. In the same way, the Ambarzumyan Theorems for Neumann as well as Dirichlet boundary conditions as given in [10, Theorems 1.3 and 5.1] can also be proved for \( L^1 \) potentials. Furthermore, the above method can also be used to show Theorem 1.1 by reducing the periodic problem to a Dirichlet problem by a translation of the first nodal length, as in [8].

In fact, for general linear separated boundary problems (3.1),
\[
\mu_n^{1/p} = n_{\alpha\beta}\hat{\pi}
\]
\[
+ \frac{(\text{CT}_p(\beta))^{(p-1)} - (\text{CT}_p(\alpha))^{(p-1)}}{(n_{\alpha\beta}\hat{\pi})^{p-1}} + \frac{1}{p(n_{\alpha\beta}\hat{\pi})^{p-1}} \int q(x) dx + o\left(\frac{1}{n^{p-1}}\right),
\]
where
\[
n_{\alpha\beta} = \begin{cases} 
  n & \text{if } \alpha = \beta = 0 \\
  n - 1/2 & \text{if } \alpha > 0 = \beta \text{ or } \beta > 0 = \alpha \\
  n - 1 & \alpha, \beta > 0 
\end{cases}
\]
This is because, after an integration of (2.2),
\[
\theta_n(1) - \theta_n(0) = 1 - \frac{1}{\mu_n} \int_0^1 q(x)|S_p(\mu_n^{1/p}\theta(x))|^p dx + o\left(\frac{1}{\mu_n}\right).
\]
By (3.2), if \( \alpha = 0 \), then \( \theta_n(0) = 0 \). Similarly \( \theta_n(1) = 0 \) if \( \beta = 0 \). Now, let \( y = CT_p^{-1}(x) \). Then \( x = CT_p(y) \) and hence
\[
y' = \frac{-|x|^{p-2}}{1 + |x|^p} = -|x|^{p-2}(1 + O(|x|^p)),
\]
Inverse Nodal Problems for Periodic $p$-Laplacian

when $|x|$ is sufficiently small. Since $y(0) = \frac{\pi}{2}$, we have

$$y(x) = \frac{\pi}{2} - \frac{x^{(p-1)}}{p} + O(x^{2(p-1)}).$$

Therefore, when $n$ is sufficiently large,

$$\theta_n(0) = \frac{\pi}{2} + \frac{\left(CT_p(\alpha)\right)^{(p-1)}}{(p-1)\mu_n^{(p-1)/p}} + O\left(\mu_n^{-1}\right).$$

Similarly, when $\beta \neq 0$,

$$\theta_n(1) = \frac{\pi}{2} + \frac{\left(CT_p(\beta)\right)^{(p-1)}}{(p-1)\mu_n^{(p-1)/p}} + O\left(\mu_n^{-1}\right).$$

Hence (3.4) is valid. Furthermore, $F_n$ converges to $q$ pointwise and in $L^1(0, 1)$, where

$$F_n(x) := p(n_{\alpha\beta}\pi)^p\left[\frac{n_{\alpha\beta} + \left(\frac{CT_p(\alpha)}{(n_{\alpha\beta}\pi)^{p-1}}\right)\theta_j^{(n)}}{\theta_j^{(n)}} - 1\right] + \int_0^1 q(t) \, dt.$$

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