ZEROS OF A QUASI-MODULAR FORM OF WEIGHT 2 FOR $\Gamma_0^+(N)$

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Abstract. Basraoui and Sebbar showed that the Eisenstein series $E_2$ has infinitely many $SL_2(\mathbb{Z})$-inequivalent zeros in the upper half-plane $\mathbb{H}$, yet none in the standard fundamental domain $\mathfrak{F}$. They also found infinitely many such regions containing a zero of $E_2$ and infinitely many regions which do not have any zeros of $E_2$. In this paper we study the zeros of the quasi-modular form $E_2(z) + NE_2(Nz)$ of weight 2 for $\Gamma_0^+(N)$.

1. INTRODUCTION AND PRELIMINARIES

It is well known by the the Valence formula [12, Section 1.3, Proposition 2] that every nonzero modular form has finitely many $SL_2(\mathbb{Z})$-inequivalent zeros in the upper half-plane $\mathbb{H}$. Several authors investigated the zeros of special modular forms for $SL_2(\mathbb{Z})$ (for example, see [3, 5, 9, 4]). It has been proved that for an even integral weight $k$ the Eisenstein series $E_k$ for $SL_2(\mathbb{Z})$, the zeros of $E_k$ in the fundamental domain of the modular group $SL_2(\mathbb{Z})$ lie in the arc of the unit circle for $4 \leq k \leq 26$ by Wohlfahrt [11] and for every $k > 2$, by Rankin and Swinnerton-Dyer [8] later. Rankin [7] generalized this result to a certain class of Poincaré series for $SL_2(\mathbb{Z})$.

For higher level cases, let $\Gamma_0^+(N)$ denote the group generated by the Hecke congruence group $\Gamma_0(N)$ and the Fricke involution $w_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Shigezumi [6] investigated the zeros of the Eisenstein series for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$. Recently Basraoui and Sebbar [1] investigated some properties of zeros of the Eisenstein series $E_2$ for $SL_2(\mathbb{Z})$ which is a quasi-modular form. They showed that there are infinitely many...
inequivalent zeros of $E_2$ in the half strip $\mathcal{S} := \{ \tau \in \mathbb{H} | -1/2 < \text{Re}(\tau) \leq 1/2 \}$ and proved that the fundamental domain $\mathfrak{F}$ for $\text{SL}_2(\mathbb{Z})$ and infinitely many of its conjugates in $\mathfrak{F}$ contain no zeros of $E_2$, while there are infinitely many conjugates of $\mathfrak{F}$ in $\mathcal{S}$ which contain zeros of $E_2$. This is a different phenomenon from the cases for modular forms.

In this paper, by applying the arguments in [1] we study the zeros of the quasi-modular form $E_2(z) + NE_2(Nz)$ of weight 2 for $\Gamma_0^+(N)$, whose definition is given in Definition 1.1. In particular, we show how to take care of the parts related with the Fricke involution while the proofs in [1] deal with $\text{SL}_2(\mathbb{Z})$.

Throughout this paper, we let $z = x + iy$ with $x, y > 0 \in \mathbb{R}$ and denote $\Gamma_0(N)$ or $\Gamma_0^+(N)$ by $\Gamma$.

**Definition 1.1.** ([12, page 58]). For a positive even integer $k$, an almost holomorphic modular form of weight $k$ and depth $\leq M$ for $\Gamma$ is a holomorphic function $F(z)$ on $\mathbb{H}$ such that

$$F(\frac{az + b}{cz + d}) = (\text{det}\gamma)^{-k/2} (cz + d)^k F(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and the growth condition that it has the form

$$F(z) = \sum_{m=0}^{M} f_m(z) (-4\pi y)^{-m},$$

(1) where $f_0(z), \ldots, f_M(z)$ are holomorphic on $\mathbb{H}$ for some nonnegative integer $M$ (which is necessarily at most $k/2$).

The constant term, $f_0(z)$ of such a $F$ is called a quasi-modular form of weight $k$ for $\Gamma$. We let $\tilde{M}_k(\Gamma)$ be the $\mathbb{C}$-linear space of quasi-modular forms of weight $k$ for $\Gamma$. Then the space $\tilde{M}_k(\Gamma) = \bigoplus \tilde{M}_k(\Gamma')$ is a graded ring. Note that as mentioned in [12, page 58], a direct definition of a quasi-modular form of weight $k$ and depth $\leq M$ on $\Gamma$ can be given as a holomorphic function $f$ on $\mathbb{H}$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the function $(\text{det}\gamma)^{k/2} (cz + d)^{-k} f \left( \frac{az+b}{cz+d} \right)$ is a polynomial of degree $\leq M$ in $\frac{c}{cz+d}$.

Indeed, if we choose a holomorphic function $\phi$ on $\mathbb{H}$ such that the function $\phi^*(z) := \phi(z) - 1/(4\pi y)$ satisfies the following,

$$\phi^*(\gamma z) = (\text{det}\gamma)^{-1}(cz + d)^2 \phi^*(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

where $z = x + iy$, then clearly $\phi$ is a quasi-modular form of weight 2 for $\Gamma$. We can show that every quasi-modular form of weight $k$ for $\Gamma$ is presented as a polynomial of a quasi-modular form $\phi$ of weight 2 with coefficients of modular forms as follows:
Proposition 1.2. ([12, page 59]). For a positive even integer $k$ and an integer $r$ such that $0 \leq r \leq k/2$, let $M_{k-2r}(\Gamma)$ be the space of modular forms of weight $k - 2r$ for $\Gamma$ where $\Gamma$ is $\Gamma_0(N)$ or $\Gamma^+_0(N)$. A quasi-modular form of weight $k$ for $\Gamma$ is an element in the ring $\bigoplus_{r=0}^{k/2} M_{k-2r}(\Gamma) \cdot \phi^r$, where $\phi$ is a holomorphic function on $\mathbb{H}$ satisfying the condition (1).

We recall that the Eisenstein series $E_2(z)$ is written as

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

where $\sigma_1(n) = \sum_{1 \leq d \mid n} d$.

Then this is a quasi-modular form of weight 2 for $SL_2(\mathbb{Z})$ and it satisfies that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$E_2 \left( \frac{az + b}{cz + d} \right) = (cz + d)^2 E_2(z) - \frac{6i}{\pi} c(cz + d).$$

(This is by normalization of [12, Section 2.3, Eq. (17) and (19)].)

For convenience, we define the slash operator $f \mapsto f_{2\gamma}$ by

$$(f_{2\gamma})(z) = (\det \gamma)(cz + d)^{-2} f \left( \frac{az + b}{cz + d} \right), \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL^+_2(\mathbb{R}),$$

and so we have the definition,

$$(f(g)_{2\gamma})(z) = (\det \gamma)(cz + d)^{-2} f((g(z))), \text{ for a function } g : \mathbb{H} \to \mathbb{H}.$$}

We now prove that $E_2(z) + NE_2(Nz)$ is a quasi-modular form of weight 2 for $\Gamma^+_0(N)$ and calculate some special values of $E_2(z) + NE_2(Nz)$ which will be needed later.

**Proposition 1.3.**

(1) $E_2(z) + NE_2(Nz)$ is a quasi-modular form of weight 2 on $\Gamma^+_0(N)$.

(2) $E_2(z) - NE_2(Nz)$ is a modular form of weight 2 on $\Gamma_0(N)$.

**Proof.** We let

$$E^*_2(z) := E_2(z) - \frac{3}{\pi y}.$$

Then $E^*_2$ is invariant under the slash operator $|2$ for all $\gamma \in SL_2(\mathbb{Z})$. 
(1) Let $E(z) = E_2(z) + NE_2(Nz)$. Then

$$E(z) = E^*_2(z) + \frac{3}{\pi y} + N\left( E^*_2(Nz) + \frac{3}{\pi Nz} \right).$$

(3) Then

$$E(z) = E^*_2(z) + NE^*_2(Nz) + \frac{6}{\pi y}.$$ 

Hence

(4) 

$$E(z) - \frac{6}{\pi y} = E^*_2(z) + NE^*_2(Nz).$$

Let $g(z) = Nz$. Considering $E^*_2(Nz) = E^*_2(g(z))$, we have that for any $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$,

$$((E^*_2 + NE^*_2(g))|_2 \gamma)(z) = E^*_2(N\gamma z)(cNz + d)^{-2}$$

(5) 

$$= E^*_2\left( \frac{a(Nz) + bN}{c(Nz) + d} \right)(cNz + d)^{-2}$$

$$= (E^*_2|_2 \gamma')(Nz) = E^*_2(Nz) = E^*_2(g(z)),$$

where $\gamma' = \begin{pmatrix} a & bN \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. (Note that the last equality follows from the fact that $E^*_2$ is invariant under the slash operator $|_2$.)

Hence this implies that for all $\gamma \in \Gamma_0(N)$,

$$((E^*_2 + NE^*_2(g))|_2 \gamma)(z) = E^*_2(z) + NE^*_2(Nz).$$

Now for $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, we have that

$$((E^*_2 + NE^*_2(g))|_{2w_N})(z) = (\sqrt{Nz})^{-2} E^*_2\left( \frac{-1}{Nz} \right) + NE^*_2\left( \frac{-1}{z} \right)$$

(6) 

$$= N^{-1}z^{-2}E^*_2\left( \frac{-1}{Nz} \right) + z^{-2}E^*_2\left( \frac{-1}{z} \right)$$

$$= N(Nz)^{-2}E^*_2\left( \frac{-1}{Nz} \right) + z^{-2}E^*_2\left( \frac{-1}{z} \right)$$

$$= E^*_2(z) + NE^*_2(Nz).$$
Note that the last inequality follows from the modularity under \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Hence we have shown that for \(g(z) = Nz\), \(((E_2^* + NE_2^*(g))(2\gamma))(z) = (E_2^*(z) + NE_2^*(Nz))\), for all \(\gamma \in \Gamma^+_0(N)\). This fact together with two conditions (1) and (4) implies that \(E(z)\) is a quasi-modular form of weight 2 on \(\Gamma^+_0(N)\).

(2) Let \(g(z) = Nz\). For all \(\gamma \in \Gamma_0(N)\), we have

\[
((E_2 - NE_2(g))(2\gamma))(z) = ((E_2^* - NE_2^*(g))(2\gamma))(z) \\
= E_2^*(z) - NE_2^*(Nz) \\
= E_2(z) - NE_2(Nz).
\]

Also, we note from (2) that for each \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\),

\[
E_2\left(\frac{az + b}{cz + d}\right) (cz + d)^{-2} = E_2(z) - \frac{6i}{\pi} \frac{c}{cz + d}. 
\]

Let \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) and let \(s := \gamma \infty = \frac{a}{c}\). Then \(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\) \(\gamma = \gamma'U\) for some \(\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \text{SL}_2(\mathbb{Z})\) and \(U = \begin{pmatrix} x & y \\ 0 & w_s \end{pmatrix} \in M_2(\mathbb{Z})\). So \(N = xw_s\), \(c = c'x\) and \(d = c'y + d'w_s\). Hence \(N/w_s = c/c'\). Therefore, we have

\[
E_2(N\gamma z) = E_2(\gamma'Uz) \\
= (c'Uz + d')^2 E_2(Uz) - \frac{6c'i}{\pi} (c'Uz + d') \\
= \frac{(cz + d)^2 E_2(Uz)}{w_s^2} - \frac{6c'i(cz + d)}{\pi w_s}. 
\]

Hence,

\[
E_2(N\gamma z) (cz + d)^{-2} = \frac{E_2(Uz)}{w_s^2} - \frac{6c'i}{\pi w_s (cz + d)} \\
= \frac{E_2(Uz)}{w_s^2} - \frac{6ci}{N\pi (cz + d)}. 
\]
\[(E_2 - NE_2(g))|_{2\gamma}(z) = (E_2(\gamma z) - NE_2(N\gamma z))(cz + d)^{-2}\]

\[(9)\]

\[E_2(z) - \frac{6ci}{\pi (cz + d)} - \frac{N}{w_s^2}E_2(Uz) + \frac{6ci}{\pi (cz + d)}\]

\[= E_2(z) - \frac{N}{w_s^2}E_2(Uz)\]

and this implies that \(E_2(z) - NE_2(Nz)\) is holomorphic at the cusp \(s\). Consequently \(E_2(z) - NE_2(Nz)\) is a modular form of weight 2 on \(\Gamma_0(N)\).

Throughout this paper, as in the proof of Proposition 1.3 we let

\[E(z) := E_2(z) + NE_2(Nz)\]

for \(z \in \mathbb{H}\). Then for \(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\), we can easily show by (2) that

\[(10)\]

\[E\left(\frac{az + b}{cz + d}\right) = (cz + d)^2E(z) - \frac{12i}{\pi}c(cz + d).\]

Note that \(\rho_2 := e^{i(3\pi/4)}/\sqrt{2}\) is an elliptic point of nonzero modular functions of weight \(k\) for \(\Gamma_0^+(2)\) by [6, Proposition 3.1] and \(\rho_3 := e^{i(2\pi/3)}/\sqrt{3}\) is an elliptic point for \(\Gamma_0^+(3)\) by [6, Proposition 4.3].

**Lemma 1.4.**

(a) \(E(\rho_2) = \frac{12}{\pi}\) for \(N = 2\).

(b) \(E(\rho_3) = \frac{12\sqrt{3}}{\pi}\) for \(N = 3\).

**Proof.** Note that for \(\tau \in \mathbb{H}\),

\[E\left(-\frac{1}{N\tau}\right) = E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right)\]

\[= E_2\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right) + NE_2\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right)\]

\[= (N\tau)^2E_2(N\tau) + \frac{6}{\pi ti}(N\tau) + NE_2\left(\frac{1}{\tau}\right)\] by (2)

\[= \tau^2NE(\tau) + \frac{12N}{\pi ti}\tau.\]

(a) By (11), for \(\tau = \rho_2 = e^{i(3\pi/4)}/\sqrt{2}\) with \(N = 2\),
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(12) \quad E\left(-\frac{1}{2\rho_2}\right) = -iE(\rho_2) + \frac{12}{\pi i}(-1 + i).

Now since $\alpha_2 w_2 \rho_2 = \rho_2$ for $\alpha_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \in \Gamma_0(2)$ and $w_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$, we get from (10) and (12):

\[ E(\rho_2) = E((\alpha_2 w_2)\rho_2) = E(\alpha_2(w_2 \rho_2)) \]
\[ = (-2w_2\rho_2 + 1)^2 E(w_2\rho_2) + \frac{24}{\pi i}(-2w_2\rho_2 + 1) \]
\[ = \left(\frac{1}{\rho_2} + 1\right)^2 E\left(-\frac{1}{2\rho_2}\right) - \frac{24}{\pi i}\left(\frac{1}{\rho_2} + 1\right) \]
\[ = iE(\rho_2) + \frac{12}{\pi i}(1 + i) \text{ by (12).} \]

Hence we solve $E(\rho_2) = iE(\rho_2) + \frac{12}{\pi i}(1 + i)$ for $E(\rho_2)$ and we get

\[ E(\rho_2) = \frac{12}{\pi}. \]

(b) Similarly, with $\rho_3 = e^{i(5\pi/6)/\sqrt{3}}$ and $N = 3$, we have from (11) that

(13) \quad E\left(-\frac{1}{3\rho_3}\right) = \left(\frac{1 - \sqrt{3}i}{2}\right)E(\rho_3) + \frac{6}{\pi i}(-3 + \sqrt{3}i).

And since $\alpha_3 w_3 \rho_3 = \rho_3$ for $\alpha_3 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ and $w_3 = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$, we have that

\[ E(\rho_3) = E((\alpha_3 w_3)\rho_3) = E(\alpha_3(w_3 \rho_3)) \]
\[ = (-3w_3\rho_3 + 1)^2 E(w_3\rho_3) + \frac{36}{\pi i}(-3w_3\rho_3 + 1) \]
\[ = \left(\frac{1}{\rho_3} + 1\right)^2 E\left(-\frac{1}{3\rho_3}\right) - \frac{36}{\pi i}\left(\frac{1}{\rho_3} + 1\right) \]
\[ = \left(\frac{1 + \sqrt{3}i}{2}\right)E(\rho_3) + \frac{6}{\pi i}(3 + \sqrt{3}i) \text{ by (13).} \]

So we solve $E(\rho_3) = \left(\frac{1 + \sqrt{3}i}{2}\right)E(\rho_3) + \frac{6}{\pi i}(3 + \sqrt{3}i)$ for $E(\rho_3)$ and get

\[ E(\rho_3) = \frac{12\sqrt{3}}{\pi}. \]

\[ \square \]
2. Zeros of \( E \) for \( \Gamma_0^+ (N) \)

In this section we study the zeros of \( E \) for \( \Gamma_0^+ (N) \), where \( E(z) = E_2(z) + NE_2(Nz) \).

**Proposition 2.1.** For a positive integer \( N \), the quasi-modular form \( E \) for \( \Gamma_0^+ (N) \) has a unique zero \( \tau_0 \) on the imaginary axis. And for \( N = 2, 3 \), \( E \) for \( \Gamma_0^+ (N) \) has a zero \( \tau_1 \) on the axis \( \text{Re}(z) = \frac{1}{2} \).

**Proof.** This uses the proof of [1, Proposition 3.1] for \( E_2 \).

For \( \tau = iy \), since \( E_2(\tau) \) is real and increasing on \((0, \infty)\) by definition of \( E_2 \), \( E(\tau) \) is also real and increasing on \((0, \infty)\).

Also since \( \lim_{y \to 0} E_2(iy) = -\infty \) and \( \lim_{y \to \infty} E_2(iy) = 1 \),

\[
\lim_{y \to 0} E(iy) = -\infty \quad \text{and} \quad \lim_{y \to \infty} E(iy) = 1 + N > 1. 
\]

Since \( E(iy) \) is continuous and increasing, this implies that \( E \) has a unique zero, say \( \tau_0 \) on the purely imaginary axis.

Note that \( E_2(\tau) \) is real for \( \tau = \frac{1}{2} + iy \), \( y > 0 \), and \( \lim_{y \to 0} E_2(\frac{1}{2} + iy) = -\infty \). If \( N \) is even, then

\[
\lim_{y \to 0} E \left( \frac{1}{2} + iy \right) = \lim_{y \to 0} \left( E_2 \left( \frac{1}{2} + iy \right) + NE_2(Niy) \right) = -\infty,
\]

and if \( N \) is odd, then

\[
\lim_{y \to 0} E \left( \frac{1}{2} + iy \right) = \lim_{y \to 0} \left( E_2 \left( \frac{1}{2} + iy \right) + NE_2 \left( \frac{1}{2} + Niy \right) \right) = -\infty.
\]

If \( N = 2 \), by Lemma 1.4 (1), \( E(\rho_2) = E(\rho_2 + 1) = \frac{12\pi^{2}}{N} > 0 \), hence we conclude that there exists a zero \( \tau_1 \) of real part \( 1/2 \) and whose imaginary part is less than \( 1/2 \).

If \( N = 3 \), by Lemma 1.4 (2), \( E(\rho_3) = E(\rho_3 + 1) = \frac{12\pi^{2}}{N} > 0 \), hence we conclude that there exists a zero \( \tau_1 \) of real part \( 1/2 \) and whose imaginary part is less than \( 1/(2\sqrt{3}) \).

**Proposition 2.2.** For each integer \( N \geq 2 \), two zeros of \( E \) are \( \Gamma_0^+ (N) \)-equivalent if and only if one is a translation of the other by an integer.

**Proof.** Suppose that \( z_1 \) and \( z_2 \) are any two zeros of \( E \) in \( \mathbb{H} \) that are equivalent modulo \( \Gamma_0^+ (N) \), i.e. \( z_1 = \alpha z_2 \) for some \( \alpha \in \Gamma_0^+ (N) \).

If \( \alpha \in \Gamma_0(N) \), \( \alpha \) must be a translation as in the proof of [1, Proposition 3.3].

If \( \alpha = \gamma w_N \), where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \) and \( w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \), then we have from (10) and (11) that
0 = E(z_1) = E(γ(w_Nz_2)) = (cw_Nz_2 + d)^2 E(w_Nz_2) + \frac{12\pi i}{\pi i} (cw_Nz_2 + d) and
\[
E(w_Nz_2) = \frac{12N}{\pi i} z_2 + Nz_2^2 E(z_2) = \frac{12N}{\pi i} z_2.
\]

Hence 0 = (cw_Nz_2 + d)^2 \frac{12N}{\pi i} z_2 + \frac{12\pi i}{\pi i} (cw_Nz_2 + d) implies that cw_Nz_2 + d = 0 or (cw_Nz_2 + d)Nz_2 + c = 0. Note that w_Nz_2 ∈ \mathbb{H} implies that cw_Nz_2 + d ≠ 0, since γ ∈ Γ_0(N). So 0 = (cw_Nz_2 + d)Nz_2 + c = (Nz_2)Nz_2 + c = dNz_2. Then d = 0 and -bc = 1, so c = ±1, and then γ ∉ Γ_0(N), which is a contradiction.

The invariance of \( E \) under translation proves the converse.

\[\Box\]

**Corollary 2.3.** For each integer \( N \geq 2 \), no two distinct zeros of \( E \) for \( \Gamma_0^+(N) \) in the half-strip \( \mathcal{S} = \{ \tau \in \mathbb{H} : -\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2} \} \) are equivalent modulo \( \Gamma_0^+(N) \).

**Theorem 2.4.** For each integer \( N \geq 2 \), the quasi-modular form \( E \) for \( \Gamma_0^+(N) \) has infinitely many \( \Gamma_0^+(N) \)-inequivalent zeros in the half-strip \( \mathcal{S} \).

**Proof.** By [10, Proposition 5.3] with \( f = NE_2(Nz) - E_2 \) and \( \phi_0 = 2E_2 \) for \( E = f + \phi_0 \), \( E \) has infinitely many zeros that are inequivalent relative to \( \Gamma_0(N) \), so to \( \Gamma_0^+(N) \). Hence since it is invariant under translation, the theorem holds.

Next, we are interested in \( \Delta_N^+ \) for \( N = 2, 3 \) defined as in [2, eq.(10)]:

\[
(15) \quad \Delta_N^+ = (\eta(z)\eta(Nz))^\delta, \quad \text{where } \delta = \begin{cases} 8, & \text{if } N = 2 \\ 12, & \text{if } N = 3. \end{cases}
\]

**Corollary 2.5.** \( \Delta_N^+ \) has infinitely many critical points for \( N = 2, 3 \).

**Proof.** Note that for \( f \in M_k(\Gamma_0^+(N)) \),
\[
\partial_k f = \theta f - \frac{kE}{24} f \in M_{k+2}(\Gamma_0^+(N)).
\]
By (15), \( \Delta_2^+ = (\eta(z)\eta(2z))^8 \) and \( \Delta_3^+ = q + \mathcal{O}(q^2) \in S_6(\Gamma_0^+(2)) \). Hence,
\[
\partial_k \Delta_2^+ = \theta \Delta_2^+ - \frac{8E}{24} \Delta_2^+ = \mathcal{O}(q) - \frac{8E}{24} \mathcal{O}(q) = \mathcal{O}(q) \in S_10(\Gamma_0^+(2)).
\]

Since \( \dim(S_{10}(\Gamma_0^+(2))) = \left\lfloor \frac{10}{8} \right\rfloor - 1 = 0 \), we have that \( \partial_k \Delta_2^+ = 0 \), so \( \theta \Delta_2^+ = \frac{8E}{24} \Delta_2^+ \) and \( E = 3 \left( \frac{\theta \Delta_2^+}{\Delta_2^+} \right) \). Therefore our assertion for \( N = 2 \) follows from Theorem 2.4.
Again, by (15), \( \Delta^+_3 = (\eta(z)\eta(3z))^{12} \) and \( \Delta^+_3 = q^2 + \mathcal{O}(q^3) \in S_{12}(\Gamma^+_0(3)) \). Hence,
\[
\partial_{12}\Delta^+_3 = \theta\Delta^+_3 - \frac{12E}{24}\Delta^+_3
\]
\[
= \mathcal{O}(q^2) - \frac{12E}{24}\mathcal{O}(q^2)
\]
\[
= \mathcal{O}(q^2) \in S_{14}(\Gamma^+_0(3))
\]. Hence,
\[
\partial_{12}\Delta^+_3 = \theta\Delta^+_3 - \frac{12E}{24}\Delta^+_3
\]
\[
= \mathcal{O}(q^2) - \frac{12E}{24}\mathcal{O}(q^2)
\]
\[
= \mathcal{O}(q^2) \in S_{14}(\Gamma^+_0(3))
\].

Since \( \dim(S_{14}(\Gamma^+_0(3))) = \left[ \frac{14}{6} \right] - 1 = 1 \), there is no a nonzero modular form with a Fourier expansion at \( \infty \) starting \( q^n \) for \( n > 1 \), which implies that \( \partial_{12}\Delta^+_3 = 0 \). So \( \theta\Delta^+_3 = \frac{12E}{24}\Delta^+_3 \) and \( E = 2\left( \frac{\theta\Delta^+_3}{\Delta^+_3} \right) \). Therefore our assertion for \( N = 3 \) follows from Theorem 2.4.

\[\text{3. DISTRIBUTION OF THE ZEROS OF } E \text{ FOR } \Gamma^+_0(2)\]

Note that a fundamental domain for \( \Gamma^+_0(2) \) is given by
\[
\mathfrak{F}^+(2) := \{|z| \geq 1/\sqrt{2}, -1/2 \leq \text{Re}(z) \leq 0\} \cup \{|z| > 1/\sqrt{2}, 0 \leq \text{Re}(z) < 1/2\}.
\]
(Refer to [6, p. 694].)

We consider fundamental regions within the half-strip that contains zeros of \( E \) and fundamental regions that do not contain any zeros of \( E \).

**Theorem 3.1.** There exists a positive integer \( c_0 \) such that for all odd integers \( c \) with \( |c| \geq c_0 \), there exists a fundamental domain with a vertex at \( \frac{c-1}{2c} \) containing a zero of \( E \). Therefore, there exist infinitely many fundamental domains within the half-strip that contains zeros of \( E \).

**Proof.** By generalizing the idea of the proof of [1, Theorem 4.1], let \( \tau_0 \) be the unique zero of \( E \) on the imaginary axis and let \( \alpha = \left( \begin{array}{cc} t & u \\ v & w \end{array} \right) \in \Gamma_0(2) \), where \( t \neq 0 \). Then,
\[
E(\tau_0) = 0 = E(\alpha^{-1}(\alpha \tau_0)) = (-v\alpha \tau_0 + t)^2E(\alpha \tau_0) - \frac{12i}{\pi}(-v)(-v\alpha \tau_0 + t).
\]
This is true if and only if
\[
\frac{E(\alpha \tau_0)}{\alpha \tau_0 E(\alpha \tau_0) + \frac{12i}{\pi}} = \frac{v}{i}.
\]
(16)

Note that \( \tau_0 \in w_2\mathfrak{F}^+(2) \). In fact, from (11) we have that
\[
E\left( -\frac{1}{2}, \frac{-1}{\sqrt{2}} \right) = \left( \frac{i}{\sqrt{2}} \right)^2 2E\left( \frac{i}{\sqrt{2}} \right) + \frac{24}{\pi i} \cdot \frac{i}{\sqrt{2}}.
\]
which implies that
\[ E \left( \frac{i}{\sqrt{2}} \right) = \frac{6\sqrt{2}}{\pi} > 0. \]
Since \( E \) is strictly increasing on \((0, \infty)\) along the imaginary axis, \( \tau_0 = iy \) is below \( \frac{i}{\sqrt{2}} \), therefore \( 0 < y < \frac{1}{\sqrt{2}} \). Note that
\[ \tau_0 \in w_2 \mathcal{F}^+ (2) \iff w_2 \tau_0 = \frac{1}{2iy} = \frac{i}{2y} \in \mathcal{F}^+ (2) \iff \text{Im}(w_2 \tau_0) = \frac{1}{2y} > \frac{1}{\sqrt{2}} \iff 0 < y < \frac{1}{\sqrt{2}}. \]

Hence, when
\[ f(z) = \frac{E(z)}{zE(z) + \frac{12}{\pi}} \]
and \( \alpha = S_{-2} := \left( 1 0 \right) \begin{pmatrix} \frac{1}{-2} & 0 \\ 1 & 1 \end{pmatrix} \), this implies that \( f \) maps a neighborhood \( D_0 \) of \( S_{-2} \tau_0 \), which can be chosen to be in the interior of \( S_{-2}w_2 \mathcal{F}^+ (2) \) onto a neighborhood \( U_0 \) of \(-2\).

There exists a positive integer \( c_0 \) such that for all integers \( c \) such that \(|c| \geq c_0\), \(-2 - \frac{2}{c} \in U_0\). For each odd integer \(|c| \geq c_0\), let \( z_c \in D_0 \) such that \( f(z_c) = -2 - \frac{2}{c} \).

Therefore, if \( \gamma_c = \left( \begin{array}{c} c \\ 2c + 2 \end{array} \right) \begin{pmatrix} \frac{c-1}{c} \\ -1 \\ c \end{pmatrix} \in \Gamma_0 (2) \subset \Gamma_0^+ (2) \), then since
\[ \frac{E(\gamma_c^{-1}(\gamma_c z_c))}{\gamma_c^{-1}(\gamma_c z_c)E(\gamma_c^{-1}(\gamma_c z_c)) + \frac{12}{\pi}} = \frac{E(z_c)}{z_cE(z_c) + \frac{12}{\pi}}, \]
recalling (16), \( \gamma_c z_c \) is a zero of \( E \) belonging to \( \gamma_c S_{-2}w_2 \mathcal{F}^+ (2) \). For all odd integers \( c \) such that \(|c| \geq c_0\),
\[ \gamma_c S_{-2}w_2 = \left( \begin{array}{c} c - 1 \\ 2c \end{array} \right) \begin{pmatrix} -1 \\ -2 \end{pmatrix} \in \Gamma_0^+ (2), \]
and \( \gamma_c S_{-2}w_2 (\infty) = \frac{c-1}{2c} \). Hence \( \gamma_c S_{-2}w_2 \mathcal{F}^+ (2) \) is the fundamental domain which has a vertex at the cusp \( \frac{c-1}{2c} \).

**Proposition 3.2.** The Eisenstein series \( E \) for \( \Gamma_0^+ (2) \) has no zeros in the fundamental domain \( \mathcal{F}^+ (2) \) for \( \Gamma_0^+ (2) \).

**Proof.** Let \( \tau_0 = iy_0 \) be the unique zero of \( E \) on the imaginary axis. Then, by (11), we have that
\[ E \left( \frac{-1}{2 \cdot iy_0} \right) = (iy_0)^2 \cdot E(iy_0) + \frac{24}{\pi i} \cdot iy_0 = \frac{24}{\pi} y_0 < 3. \]
The last inequality follows from the following: Since \( \lim_{y \to \infty} E(iy) = 3 \) by (14), and \( E \) is strictly increasing on \((0, \infty)\) along the imaginary axis, we have that \( E \left( -\frac{1}{2} \cdot iy_0 \right) = \frac{24}{\pi} y_0 < 3. \)

This inequality implies that \( y_0 < \frac{\pi}{8}. \) If \( \tau = x + iy \in \mathfrak{H}^+(2) \) is a zero of \( E \), then \( y = \text{Im}(\tau) > \frac{1}{2} > \frac{\pi}{8} > y_0. \) Hence we have

\[
\frac{1}{24} |3 - E(\tau)| \leq \frac{1}{24} (|1 - E_2(\tau)| + 2|1 - E_2(2\tau)|) \\
= \left| \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi in\tau} \right| + 2 \left| \sum_{n=1}^{\infty} \sigma_1(n) e^{4\pi in\tau} \right| \\
\leq \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi ny} + 2 \left( \sum_{n=1}^{\infty} \sigma_1(n) e^{-4\pi ny} \right) \\
< \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi ny_0} + 2 \left( \sum_{n=1}^{\infty} \sigma_1(n) e^{-4\pi ny_0} \right) \\
= \frac{1}{24} (3 - E(\tau_0)) = \frac{1}{8}.
\]

Hence \( |3 - E(\tau)| < 3. \) hence \( \tau \) cannot be a zero of \( E \) if \( \tau \in \mathfrak{H}^+(2). \)

Now we will find more fundamental domains which do not contain any zeros of \( E. \)

**Lemma 3.3.** For an odd positive integer \( c \), let \( S_c^+ = \left( \frac{c-1}{2c}, \frac{1-2c}{4c} \right) \in \Gamma_0(2) \! w_2. \) Then the fundamental domain \( S_c^+ \mathfrak{H}^+(2) \) is the region with the edge joining \( \frac{c-1}{2c} \) and \( S_c^+(\rho_2) \) which is an arc of the circle \( C_1(c) \) centered at \( c_1(c) = \frac{5c^2-c-1}{2c(3c+2)} \) with radius \( r_1(c) = \frac{i}{2c(3c+2)}. \) and the edge joining \( \frac{c-1}{2c} \) and \( S_c^+(\rho_2+1) \) which is an arc of the circle \( C_2(c) \) centered at \( c_2(c) = \frac{3c^2-c-1}{2c(3c+2)} \) with radius \( r_2(c) = \frac{i}{2c(3c+2)}. \)

**Proof.** Note that \( S_c^+(\infty) = \frac{1}{2} - \frac{1}{2c}. \)

\[
S_c^+(\rho_2) = \frac{13c^2 - 3c - 3}{2(13c^2 + 10c + 2)} + \frac{i}{2(13c^2 + 10c + 2)},
\]

and \( S_c^+(\rho_2+1) = \frac{5c^2 + c - 1}{2(5c^2 + 6c + 2)} + \frac{i}{2(5c^2 + 6c + 2)}. \)
Hence, from the equation of the circle centered at \( c_1(c) \in \mathbb{R} \) with radius \( r_1(c) := |c_1(c) - \frac{1}{2e}| \) passing through \( S_1^+(\rho_2) \), we find that
\[
c_1(c) = \frac{5c^2 - 3c - 1}{2c(5c + 2)} \text{ and } r_1(c) = \frac{1}{2c(5c + 2)},
\]
and similarly from the equation of the circle centered at \( c_2(c) \in \mathbb{R} \) with radius \( r_2(c) := |c_2(c) - \frac{1}{2e}| \) passing through \( S_1^+(\rho_2 + 1) \), we get that
\[
c_2(c) = \frac{3c^2 - c - 1}{2c(3c + 2)} \text{ and } r_2(c) = \frac{1}{2c(3c + 2)}.
\]

If we describe the fundamental domain \( S_1^+ \cong (2) \) more closely for better understanding, its vertices are
\[
\frac{c - 1}{2c}, S_1^+(\rho_2), \text{ and } S_1^+(\rho_2 + 1).
\]
Also since \( c \) is positive, we have that
\[
\frac{c - 1}{2c} < c_1(c) < c_2(c) < \text{Re}(S_1^+(\rho_2)) < \text{Re}(S_1^+(\rho_2 + 1))
\]
\[
\text{and } \text{Im}(S_1^+(\rho_2)) < \text{Im}(S_1^+(\rho_2 + 1)) < r_1(c) < r_2(c).
\]
Thus we have the following Figure 1.

![Figure 1. The fundamental domain \( S_1^+ \cong (2) \).](image)

**Theorem 3.4.** For each integer \( m \leq -4 \) and each odd integer \( c \geq 3 \), let
\[
S_1^+(m) = \left( \frac{c - 1}{2c}, \frac{m(c - 1) - 1}{2(cm - 1)} \right) \in \Gamma_0(2)w_2.
\]
Then $E$ has no zeros in $S_c^+(m) \setminus \mathbb{S}^+(2)$.

In particular, there are infinitely many fundamental domains for $\Gamma_0^+(2)$ which contain no zeros of $E$.

**Proof.** Suppose there is a zero $z_0$ of $E$ in the fundamental domain $S_c^+(m) \setminus \mathbb{S}^+(2)$. Then, $S_c^+(m) \setminus \mathbb{S}^+(2)$ has a vertex at $e^{2\pi i c}$, as does $S_c^+ \setminus \mathbb{S}^+(2)$ given in Lemma 3.3. For convenience, we let

$$b = m(c - 1) - 1 \text{ and } d = 2(cm - 1), \text{ so the given } S_c^+(m) = \left(\frac{c - 1}{2c}, \frac{b}{d}\right).$$

Then, since we assume that $m \leq -4$ and $c \geq 3$, we have that $(b, d) \neq (1 - 2c, -4c - 2)$. So $S_c^+(m) \setminus \mathbb{S}^+(2)$ is an empty set. Hence, $S_c^+(m) \setminus \mathbb{S}^+(2)$ is either within the circle $C_1(c)$ or outside the circle $C_2(c)$ on $\mathbb{H}$ given in Lemma 3.3 with referring Figure 1.

Note that

$$S_c^+(m)(\rho_2) = \frac{(2c^2m^2 - 2c^2m - 2cm^2 + c^2 - 2cm + c + 2m + 1) + i}{4(cm - 1)(c(m - 1) - 1) + 2c^2},$$

and $S_c^+(m)(\rho_2 + 1) = \frac{(2c^2m^2 + 2c^2m - 2cm^2 + c^2 - 6cm - 3c + 2m + 3) + i}{4(cm - 1)(c(m + 1) - 1) + 2c^2},$

hence, since $m \leq -4$ and $c \geq 3$, we can easily show by computation using MAPLE 16 that

$$\text{Im}(S_c^+(\rho_2)) - \text{Im}(S_c^+(m)(\rho_2 + 1))$$

$$= \frac{c(m + 3)(c(m - 2) - 2)}{13c^2 + 10c + 2}(2(cm - 1)(c(m + 1) - 1) + c^2) > 0,$$

$$\text{Im}(S_c^+(m)(\rho_2)) - \text{Im}(S_c^+(m)(\rho_2 + 1))$$

$$= \frac{2c(cm - 1)}{(2(cm - 1)(c(m + 1) - 1) + c^2))(2(cm - 1)(c(m + 1) - 1) + c^2)} < 0,$$

$$\text{Re}(S_c^+(\rho_2)) - \text{Re}(S_c^+(m)(\rho_2 + 1))$$

$$= \frac{(m + 3)(c^2(5m + 3) + 2c(m - 2) - 2)}{13c^2 + 10c + 2}(2(cm - 1)(c(m + 1) - 1) + c^2) > 0,$$

$$\text{Re}(S_c^+(m)(\rho_2 + 1)) - \text{Re}(S_c^+(m)(\rho_2))$$

$$= \frac{2((cm - 1)^2 - c^2)}{(2(cm - 1)(c(m - 1) - 1) + c^2))(2(cm - 1)(c(m + 1) - 1) + c^2)} > 0,$$

$$\text{Re}(S_c^+(m)(\rho_2)) - \frac{c - 1}{2c} = \frac{-c((2m - 1) - 2)}{2c((2cm - 1)(c(m - 1) - 1) + c^2)} > 0.$$
so

\[ \text{Im}(S_c^+(m)(\rho_2)) < \text{Im}(S_c^+(m)(\rho_2 + 1)) < \text{Im}(S_c^+(\rho_2)) < \text{Im}(S_c^+(\rho_2 + 1)), \]

and

\[ \frac{c - 1}{2c} < \text{Re}(S_c^+(m)(\rho_2)) < \text{Re}(S_c^+(m)(\rho_2 + 1)) < \text{Re}(S_c^+(\rho_2)) < \text{Re}(S_c^+(\rho_2 + 1)), \]

which implies that \( S_c^+(m) \mathfrak{H}^+(2) \) is within the circle \( C_1(c) \) on \( \mathbb{H} \) with vertices \( \frac{c - 1}{2c}, S_c^+(m)(\rho_2) \) and \( S_c^+(m)(\rho_2 + 1) \) as shown in Figure 2.

![Figure 2. The fundamental domains \( S_c^+ \mathfrak{H}^+(2) \) and \( S_c^+(m) \mathfrak{H}^+(2) \).](image)

By showing that a given zero \( z_0 \) is outside \( C_2(c) \) (hence outside \( C_1(c) \) and \( S_c^+(m) \mathfrak{H}^+(2) \)), we will get a contradiction.

Note that \( S_c^+(m) = \begin{pmatrix} -b & \frac{c - 1}{c} \\ -d & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \) and \(-bc + d \cdot \frac{c - 1}{2} = 1\). So

\[ 2(S_c^+(m))^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} c & \frac{c - 1}{2} \\ d & -b \end{pmatrix}. \]

Note that \((S_c^+(m))^{-1}z_0 = 2(S_c^+(m))^{-1}z_0\). Let \( z_1 := \begin{pmatrix} c & \frac{c - 1}{2} \\ d & -b \end{pmatrix} z_0 \).

Then,
\[ E((S_c^+(m))^{-1}z_0) = E(2(S_c^+(m))^{-1}z_0) \]
\[ = z_1^2 \cdot 2 \cdot E(z_1) + \frac{24}{\pi i}z_1 \text{ by (11)} \]
\[ = z_1^2 \cdot 2 \cdot ((dz_0 - b)^2E(z_0) + \frac{12}{\pi i}d(dz_0 - b)) + \frac{24}{\pi i}z_1 \text{ by (10)} \]
\[ = \left(\frac{c_0 - c - 1}{2z_0 - b}\right)^2 \cdot 2 \cdot \frac{12}{\pi i}d(dz_0 - b) + \frac{24}{\pi i}\left(\frac{c_0 - c - 1}{d_0 - b}\right) \]
(by the fact that \(-bc + d \cdot \frac{c - 1}{2} = 1\))
\[ = \frac{24}{\pi i}c\left(c_0 - \frac{c - 1}{2}\right). \]

So we have that
\[ \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n ((S_c^+(m))^{-1}z_0)} + 2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n 2((S_c^+(m))^{-1}z_0)} \]
\[ = \frac{3}{24} - \frac{1}{24} E(2(S_c^+(m))^{-1}z_0) \]
\[ = \frac{1}{8} - \frac{1}{24} \left(\frac{24}{\pi i}\left(c_0 - \frac{c - 1}{2}\right)\right) \]
\[ = -\frac{c^2}{\pi i}\left(z_0 - \left(\frac{c - 1}{2c} + \frac{\pi i}{8c^2}\right)\right). \]

Since \((S_c^+(m))^{-1}z_0 \in \mathbb{F}^+(2), \text{ Im}((S_c^+(m))^{-1}z_0) \geq \frac{1}{2}\). Hence
\[ \left| \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n ((S_c^+(m))^{-1}z_0)} + 2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n 2((S_c^+(m))^{-1}z_0)} \right| \]
\[ \leq \sum_{n=1}^{\infty} \sigma_1(n) e^{-n\pi} + 2 \sum_{n=1}^{\infty} \sigma_1(n) e^{-2n\pi} := M. \]

Therefore, we have that
\[ \left| z_0 - \left(\frac{c - 1}{2c} + \frac{\pi i}{8c^2}\right)\right| \leq M \frac{\pi}{c^2}. \]

Let \(D_0(c)\) be the disk centered at \(c_0(c) = \frac{c - 1}{2c} + \frac{\pi i}{8c^2}\) with radius \(r_0(c) = M \frac{\pi}{c^2}\). Refer to Figure 2. Then \(z_0\) belongs to \(D_0(c)\). In order to show that \(D_0(c)\) lies outside the circle \(C_2(c)\), we show that \(|c_0(c) - c_2(c)| > r_2(c) + r_0(c)|.\)

Since the cusp \(\frac{1}{2} - \frac{1}{2c}\) and \(c_0(c)\) are on the same vertical axis,
\[ |c_2(c) - c_0(c)|^2 = r_2(c)^2 + \left(\frac{\pi}{8c^2}\right)^2. \]
So it is enough to show that
\[ r_0(c)^2 + 2r_0(c)r_2(c) < \left( \frac{\pi}{8c^2} \right)^2, \]
which is equivalent to
\[ 64 \left( M^2 + M \frac{c}{(3c + 2)\pi} \right) < 1. \]
By modifying the proof of [1, Lemma 4.3], we set
\[ q = e^{-\pi} \approx 0.04321391825. \]
Then
\[
0 < M = \sum_{n \geq 1} \sigma_1(n)q^n + 2 \sum_{n \geq 1} \sigma_1(n)q^{2n} \\
= \sum_{n \geq 1} \frac{nq^n}{1 - q^n} + 2 \sum_{n \geq 1} \frac{nq^{2n}}{1 - q^{2n}} \quad \text{(as in the proof of [1, Lemma 4.3])} \\
\leq \frac{1}{1 - q} \sum_{n \geq 1} nq^n + \frac{1}{1 - q^2} \sum_{n \geq 1} 2nq^{2n} \\
\leq \frac{q}{(1 - q)^3} + 2 \frac{q^2}{(1 - q^2)^3} \\
\approx 0.05309361050.
\]
Since \( \frac{c}{(3c + 2)} \leq \frac{1}{3} \) for all \( c \geq 3 > 1 \), we have that
\[ 64 \left( M^2 + M \frac{c}{(3c + 2)\pi} \right) \leq 64 \left( M^2 + M \frac{1}{3\pi} \right) \approx 0.5409496650 < 1. \]
Hence we have shown that \( D_0(c) \) is outside the circle \( C_2(c) \). This completes the proof.

Remark 3.5. We note that Theorem 3.4 gives a more general and explicit description of regions comparing from the results in [1]. In particular, we show how to take care of the parts related with the Fricke involution while the proofs in [1] deal with \( \text{SL}_2(\mathbb{Z}) \).

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